ECE 206 Advanced Calculus 2, Solutions to the Midterm Test, Spring 2015

[5] **1:** Let C be the curve of intersection of the sphere $x^2+y^2+z^2=4x$ with the cone $z=\sqrt{x^2+y^2}$. Let L be the tangent line to the curve C at the point $(1,-1,\sqrt{2})$.

(a) Find a parametric equation for the line L.

Solution: For points (x,y,z) which lie on both surfaces we have $z^2=x^2+y^2=4x-x^2-x^2$ so $2x^2+2y^2=4x$ which we can write as $(x-1)^2+y^2=1$. This circle can be given parametrically by $(x,y)=\big(1+\cos t,\sin t\big)$ and then we also need $z=\sqrt{x^2+y^2}=\sqrt{(1+\cos t)^2+(\sin t)^2}=\sqrt{1+2\cos t+\cos^2 t+\sin^2 t}=\sqrt{2+2\cos t}$. Thus the curve of intersection C is given by

$$(x,y,z)=\alpha(t)=\big(1+\cos t,\sin t,\sqrt{2+2\cos t}\big)$$

and we have

$$\alpha'(t)=\Big(-\sin t,\cos t,\frac{-\sin t}{\sqrt{2+2\cos t}}\Big).$$

We choose $t=\frac{3\pi}{2}$ to get the desired point $(x,y,z)=\alpha\big(\frac{3\pi}{2}\big)=(1,-1,\sqrt{2})$. Since $\alpha'\big(\frac{3\pi}{2}\big)=\big(1,0,\frac{1}{\sqrt{2}}\big)$, the tangent line L is given parametrically by

$$(x,y,z)=\beta(s)=\alpha\big(\tfrac{3\pi}{2}\big)+s\,\alpha'\big(\tfrac{3\pi}{2}\big)=(1,-1,\sqrt{2})+s\,\big(1,0,\tfrac{1}{\sqrt{2}}\big)=\big(1+s,-1,\sqrt{2}+\tfrac{1}{\sqrt{2}}\,s\big).$$

(b) Find the point of intersection of the line L with the xy-plane.

Solution: For the point $(x,y,z)=\big(1+s,-1,\sqrt{2}+\frac{1}{\sqrt{2}}\,s\big)$ to lie on the xy-plane, we need $0=z=\sqrt{2}+\frac{1}{\sqrt{2}}\,s$ and so we must choose $s=-2$. Thus the point of intersection is

$$(x,y,z)=\beta(-2)=(-1,-1,0).$$

[5] **2:** Consider the surface $z=f(x,y)=\dfrac{4}{2+x^4+x^2+y^2}$.

(a) Find the gradient $\nabla f(1,2)$.

Solution: We have

$$\nabla f(x,y)=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)=\left(\frac{-4(4x^3+2x)}{(2+x^4+x^2+y^2)^2},\frac{-4(2y)}{(2+x^4+x^2+y^2)^2}\right)$$

and so

$$\nabla f(1,2)=\big(-\tfrac{4\cdot 6}{8^2},-\tfrac{4\cdot 4}{8^2}\big)=\big(-\tfrac{3}{8},-\tfrac{1}{4}\big).$$

(b) Find the directional derivative $D_u f(1,2)$ when $u=\big(-\frac{4}{5},\frac{3}{5}\big)$.

Solution: The directional derivative is

$$D_u f(1,2)=\nabla f(1,2)\bullet u=\big(-\tfrac{3}{8},-\tfrac{1}{4}\big)\bullet\big(-\tfrac{4}{5},\tfrac{3}{5}\big)=\tfrac{6}{20}-\tfrac{3}{20}=\tfrac{3}{20}.$$

(c) An ant walks along the above surface above the circle $(x-2)^2+y^2=5$ moving counterclockwise (when looking down from above). Find the value of $\tan\theta$, where θ is the angle (from the horizontal) at which the ant is ascending when it is at the point $\big(1,2,\frac{1}{2}\big)$.

Solution: Since the tangent to the circle $(x-2)^2+y^2=5$ at the point $(1,2)$ is perpendicular to the radius vector $(-1,2)$ (that is the vector from the centre $(2,0)$ to the point $(1,2)$) we see that (when looking down from above) the ant is moving in the direction of the vector $v=(-2,-1)$ in the xy-plane. The directional derivative of f at $(1,2)$ with respect to v is

$$D_v f(1,2)=\nabla f(1,2)\bullet v=\big(-\tfrac{3}{8},-\tfrac{1}{4}\big)\bullet(-2,-1)=\tfrac{3}{4}+\tfrac{1}{4}=1$$

and so the angle θ at which the ant is ascending is given by

$$\tan\theta=\tfrac{D_v f(1,2)}{|v|}=\tfrac{1}{\sqrt{5}}.$$

[5] **3:** Find the total charge in the region $D=\left\{(x,y,z)\middle|\sqrt{\tfrac{1}{3}(x^2+y^2)}\le z\le\sqrt{4-x^2-y^2}\right\}$ where the charge density (charge per unit volume) is given by $f(x,y,z)=x^2$.

Solution: We use spherical coordinates $(x,y,z)=\big(r\sin\phi\cos\theta,r\sin\phi\sin\theta,r\cos\phi\big)$. Some students will see immediately that the cone $z=\sqrt{\frac{1}{3}(x^2+y^2)}$ is given in spherical coordinates by $\phi=\frac{\pi}{3}$. If you do not see this immediately, then you can verify this algebraically as follows. We have

$$x^2+y^2=(r\sin\phi\cos\theta)^2+(r\sin\phi\sin\theta)^2=r^2\sin^2\phi\,(\cos^2\theta+\sin^2\theta)=r^2\sin^2\phi$$

and so (since $r\ge 0$ and $\sin\phi\ge 0$)

$$z=\sqrt{\tfrac{1}{3}(x^2+y^2)}\iff r\cos\phi=\sqrt{\tfrac{1}{3}r^2\sin^2\phi}=\tfrac{1}{\sqrt{3}}r\sin\phi\iff\tan\phi=\sqrt{3}\iff\phi=\tfrac{\pi}{3}.$$

Thus the region D is described in spherical coordinates by $0\le r\le 2$, $0\le\phi\le\frac{\pi}{3}$ and $0\le\theta\le 2\pi$. Thus the total charge is

$$\begin{aligned}
Q&=\iiint_D x^2\,dV=\int_{r=0}^{2}\int_{\phi=0}^{\pi/3}\int_{\theta=0}^{2\pi}(r\sin\phi\cos\theta)^2\cdot r^2\sin\phi\,d\theta\,d\phi\,dr\\
&=\int_{r=0}^{2}\int_{\phi=0}^{\pi/3}\int_{\theta=0}^{2\pi}r^4\sin^3\phi\cos^2\theta\,d\theta\,d\phi\,dr=\int_{r=0}^{2}\int_{\phi=0}^{\pi/3}\pi\,r^4\sin^3\phi\,d\phi\,dr\\
&=\int_{r=0}^{2}\int_{\phi=0}^{\pi/3}\pi\,r^4(1-\cos^2\phi)\sin\phi\,d\phi\,dr=\int_{r=0}^{2}\pi\,r^4\Big[-\cos\phi+\tfrac{1}{3}\cos^3\phi\Big]_{\phi=0}^{\pi/3}dr\\
&=\int_{r=0}^{2}\pi\,r^4\Big(\big(-\tfrac{1}{2}+\tfrac{1}{24}\big)-\big(-1+\tfrac{1}{3}\big)\Big)\,dr=\int_{r=0}^{2}\pi\,r^4\cdot\tfrac{-12+1+24-8}{24}\,dr\\
&=\int_{r=0}^{2}\tfrac{5\pi}{24}\,r^4\,dr=\Big[\tfrac{\pi}{24}\,r^5\Big]_{r=0}^{2}=\tfrac{32\pi}{24}=\tfrac{4\pi}{3}\,.
\end{aligned}$$

[5] **4:** Find the work done by the force $F(x,y,z)=(2xz,1+z,x^2+y)$ acting on a small object which moves along the curve given by $(x,y,z)=\alpha(t)=\big(\sqrt{1-t^2}\,,1+t\,,\sqrt{t}\big)$ for $0\le t\le 1$ in the following two ways.

(a) Calculate the work W directly from the formula $W=\int_C F\bullet T\,dL$.

Solution: We have $F\big(\alpha(t)\big)=\Big(2\sqrt{1-t^2}\,\sqrt{t}\,,\,1+\sqrt{t}\,,\,2+t-t^2\Big)$ and $\alpha'(t)=\Big(\frac{-t}{\sqrt{1-t^2}}\,,\,1\,,\,\frac{1}{2\sqrt{t}}\Big)$ and so

$$\begin{aligned}
\int_C F\bullet T\,dL&=\int_{t=0}^{1}\Big(2\sqrt{1-t^2}\,\sqrt{t}\,,\,1+\sqrt{t}\,,\,2+t-t^2\Big)\bullet\Big(\frac{-t}{\sqrt{1-t^2}}\,,\,1\,,\,\frac{1}{2\sqrt{t}}\Big)\,dt\\
&=\int_{t=0}^{1}-2t\sqrt{t}\,+1+\sqrt{t}+\frac{1}{\sqrt{t}}+\tfrac{1}{2}\sqrt{t}-\tfrac{1}{2}t\sqrt{t}\,dt\\
&=\int_{0}^{1}1+t^{-1/2}+\tfrac{3}{2}t^{1/2}-\tfrac{5}{2}t^{3/2}\,dt\\
&=\Big[t+2\,t^{1/2}+t^{3/2}-t^{5/2}\Big]_{t=0}^{1}=1+2+1-1=3.
\end{aligned}$$

(b) Find a scalar potential for F then find the work by finding the change in potential.

Solution: By inspection, we choose $g(x,y,z)=x^2z+y+yz$ to get

$$\nabla g=\Big(\tfrac{\partial g}{\partial x},\tfrac{\partial g}{\partial y},\tfrac{\partial g}{\partial z}\Big)=(2xz,1+z,x^2+y)=F(x,y,z).$$

Then, by the Conservative Fields Theorem, we have

$$\int_C F\bullet T\,dL=\int_C\nabla g\bullet T\,dL=g\big(\alpha(1)\big)-g\big(\alpha(0)\big)=g(0,2,1)-g(1,1,0)=4-1=3.$$

[5] **5:** Find the flux of the vector field $F(x,y,z) = (x+y^2, xy+z^2, xy+yz)$ through the boundary surface of the tetrahedron with vertices at $(0,1,0)$, $(2,-1,0)$, $(2,3,0)$ and $(0,1,2)$.

Solution: By visualizing the given tetrahedron D some students wills see immediately that

$$D = \Big\{(x,y,z)\Big|0 \leq x \leq 2\,,\ 1-x \leq y \leq 1+x\,,\ 0 \leq z \leq 2-x\Big\}.$$

If you cannot see this immediately, you can argue as follows. The tetrahedron D has base triangle with vertices $(0,1,0)$, $(2,-1,0)$ and $(2,3,0)$. This base triangle is given by $\big\{(x,y)\big|0 \leq x \leq 2, 1-x \leq y \leq 1+x\big\}$. The top vertex of D is at $(0,1,2)$, directly above the base vertex $(0,1,0)$, and the top triangular face of D is the triangle through $(0,1,2)$, $(2,-1,0)$ and $(2,3,0)$. The top face lies in the plane with direction vectors $(2,-1,0)-(0,1,2) = (2,-2,-2) = 2(1,-1,-1)$ and $(2,3,0)-(0,1,2) = (2,2,-2) = 2(1,1,-1)$. This plane has normal vector $(1,-1,-1)\times(1,1,-1) = (2,0,2) = 2(1,0,1)$ so it has an equation of the form $x+z=c$. Putting in the point $(x,y,z) = (0,1,2)$ shows that $c=2$, and so the top face of D lies in the plane $x+z=2$. This shows that

$$D = \Big\{(x,y,z)\Big|0 \leq x \leq 2\,,\ 1-x \leq y \leq 1+x\,,\ 0 \leq z \leq 2-x\Big\}.$$

Also note that $\nabla \cdot F = 1+x+y$. By Stokes' Theorem, the flux through the boundary surface S of the tetrahedron D is

$$\begin{aligned}
\Phi &= \iint_S F\cdot N\ dA = \iiint_D \nabla\cdot F\ dV = \int_{x=0}^{2}\int_{y=1-x}^{1+x}\int_{z=0}^{2-x} 1+x+y\ dz\,dy\,dx \\
&= \int_{x=0}^{2}\int_{y=1-x}^{1+x} (1+x+y)(2-x)\ dy\,dx = \int_{x=0}^{2}\int_{y=1-x}^{1+x} (2+x-x^2)+(2-x)\,y\ dy\,dx \\
&= \int_{x=0}^{2}\Big[(2+x-x^2)\,y + \tfrac{1}{2}(2-x)\,y^2\Big]_{y=1-x}^{1+x} dx \\
&= \int_{x=0}^{2} (2+x-x^2)\big((1+x)-(1-x)\big) + \tfrac{1}{2}(2-x)\big((1+x)^2-(1-x)^2\big)\ dx \\
&= \int_{x=0}^{2} (2+x-x^2)(2x) + \tfrac{1}{2}(2-x)(4x)\ dx = \int_{x=0}^{2} 8x-2x^3\ dx = \Big[4x^2 - \tfrac{1}{2}x^4\Big]_{x=0}^{2} = 16-8 = 8.
\end{aligned}$$