ECE 206 Advanced Calculus 2, Solutions to the Review Problems

1: Find an implicit equation, of the form $ax+by+cz=d$, for the tangent plane to the surface given parametrically by $(x,y,z)=\sigma(s,t)=\big(\sqrt{s}\,,\,(2-s)\cos t\,,\,(2-s)\sin t\big)$ for $0\le s\le 5$ and $0\le t\le 2\pi$ at the point where $(s,t)=\big(4,\frac{\pi}{6}\big)$.

Solution: When $(s,t)=\big(4,\frac{\pi}{6}\big)$ we have

$$\sigma_s\times\sigma_t=\begin{pmatrix}\frac{1}{2\sqrt{3}}\\-\cos t\\-\sin t\end{pmatrix}\times\begin{pmatrix}0\\-(2-s)\sin t\\(2-s)\cos t\end{pmatrix}=\begin{pmatrix}\frac14\\-\frac{\sqrt3}{2}\\-\frac12\end{pmatrix}\times\begin{pmatrix}0\\1\\-\sqrt3\end{pmatrix}=\begin{pmatrix}2\\\frac{\sqrt3}{4}\\\frac14\end{pmatrix}=\tfrac14\begin{pmatrix}8\\\sqrt3\\1\end{pmatrix}$$

and so the tangent plane has normal vector $(8,\sqrt3,1)$ and its equation is of the form $8x+\sqrt3y+z=c$. Put in $(x,y,z)=\sigma\big(4,\frac{\pi}{6}\big)=(2,-\sqrt3,-1)$ to get $c=16-3-1=12$, and so the equation is $8x+\sqrt3y+2=12$.

2: Let $(x,y)=f(r,\theta)=\big(r\cos\theta\,,\,r\sin\theta\big)$, and $z=g(x,y)$, and let $h(r,\theta)=g\big(f(r,\theta)\big)$. Suppose that $h(r,\theta)=r^2e^{\sqrt3\,(\theta-\frac{\pi}{6})}$. Use the Chain Rule to find $\nabla g(\sqrt3,1)$.

Solution: Note that $(x,y)=(\sqrt3,1)$ when $(r,\theta)=\big(2,\frac{\pi}{6}\big)$ and then, by the Chain Rule,

$$\begin{aligned}Dh&=Dg\cdot Df\\ \begin{pmatrix}\frac{\partial h}{\partial r}&\frac{\partial h}{\partial\theta}\end{pmatrix}&=\begin{pmatrix}\frac{\partial g}{\partial x}&\frac{\partial g}{\partial y}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial\theta}\\\frac{\partial y}{\partial r}&\frac{\partial y}{\partial\theta}\end{pmatrix}\\ \Big(2r\,e^{\sqrt3(\theta-\frac{\pi}{6})},\,\sqrt3r^2e^{\sqrt3(\theta-\frac{\pi}{6})}\Big)&=\Big(\tfrac{\partial g}{\partial x},\tfrac{\partial g}{\partial y}\Big)\begin{pmatrix}\cos\theta&-r\sin\theta\\\sin\theta&r\cos\theta\end{pmatrix}\\ \big(4\,,\,4\sqrt3\big)&=\Big(\tfrac{\partial g}{\partial x},\tfrac{\partial g}{\partial y}\Big)\begin{pmatrix}\frac{\sqrt3}{2}&-1\\\frac12&\sqrt3\end{pmatrix}\end{aligned}$$

and so

$$\nabla g^T=\big(\tfrac{\partial g}{\partial x},\tfrac{\partial g}{\partial y}\big)=\big(4,3\sqrt3\big)\begin{pmatrix}\frac{\sqrt3}{2}&-1\\\frac12&\sqrt3\end{pmatrix}^{-1}=\big(4,4\sqrt3\big)\cdot\tfrac12\begin{pmatrix}\sqrt3&1\\-\frac12&\frac{\sqrt3}{2}\end{pmatrix}=\big(\sqrt3,5\big).$$

3: Find the total charge in the solid tetrahedron with vertices at $(0,1,0)$, $(1,0,0)$, $(3,1,0)$ and $(3,1,3)$ where the charge density (charge per unit volume) is given by $\rho(x,y,z)=y$.

Solution: Let D denote the given tetrahedron with vertices at $a=(0,1,0)$, $b=(1,0,0)$, $c=(3,1,0)$ and $d=(3,1,3)$. The top view of the tetrahedron D (the projection of D onto the xy-plane) is the triangle T with vertices $a'=(0,1)$, $b'=(1,0)$, and $c'=d'=(3,1)$, that is $T=\big\{(x,y)\big|0\le y\le1\,,\,1-y\le y\le1+2y\big\}$. The top face of the tetrahedron D has direction vectors. $u=d-a=(3,0,3)=3(1,0,1)$ and $v=d-b=(2,1,3)$ and so it has normal vector $w=(1,0,1)\times(2,1,3)=(-1,-1,1)$. Thus the equation of the top face is of the form $x+y-z=c$ and we can put in the point $(x,y,z)=a=(0,1,0$ to get $c=1$. Thus the tetrahedron is the region

$$D=\big\{(x,y,z)\big|0\le y\le1\,,\,1-y\le x\le1+2y\,,\,0\le z\le x+y-1\big\}.$$

Thus the total charge is

$$\begin{aligned}Q&=\iiint_D y\;dV=\int_{y=0}^1\int_{x=1-y}^{1+2y}\int_{z=0}^{x+y-1}y\;dz\,dx\,dy\\&=\int_{y=0}^1\int_{x=1-y}^{1+2y}y(x+y-1)\,dx\,dy=\int_{y=0}^1\Big[\tfrac12yx^2+(y^2-y)x\Big]_{x=1-y}^{1+2y}dy\\&=\int_{y=0}^1\tfrac12\big((1+2y)^2-(1-y)^2\big)+(y^2-y)\big((1+2y)-(1-y)\big)\,dy\\&=\int_{y=0}^1\tfrac12y(6y+3y^2)+(y^2-y)(3y)\,dy=\int_{y=0}^1\tfrac92y^3\;dy=\tfrac98.\end{aligned}$$

4: Find the total mass on the surface $S = \{(x,y,z)\,|\,x^2+y^2 \le 3\,,\ 2z = x^2+y^2\}$ with density (mass per unit area) $\rho(x,y,z) = x^2$.

Solution: The surface S can be given parametrically by $(x,y,z) = \sigma(r,\theta) = \big(r\cos\theta\,,\ r\sin\theta\,,\ \frac{1}{2}r^2\big)$ for $0 \le r \le \sqrt{3}$ and $0 \le \theta \le 2\pi$. We have

$$\sigma_r \times \sigma_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \\ r \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -r^2\cos\theta \\ -r^2\sin\theta \\ r \end{pmatrix} = r\begin{pmatrix} -r\cos\theta \\ r\sin\theta \\ 1 \end{pmatrix}$$

so that $\left|\sigma_r \times \sigma_\theta\right| = r\sqrt{r^2+1}$. Also note that $\rho = x^2 = (r\cos\theta)^2$. Using the substitution $u = r^2+1$ so that $du = 2r\,dr$, the mass is

$$\begin{aligned} M &= \iint_S \rho\ dA = \int_{r=0}^{\sqrt{3}} \int_{\theta=0}^{2\pi} (r\cos\theta)^2 \cdot r\sqrt{r^2+1}\ d\theta\,dr = \int_{r=0}^{\sqrt{3}} \pi r^3\sqrt{r^2+1}\,dr = \int_{u=1}^{4} \tfrac{\pi}{2}(u-1)\sqrt{u}\ du \\ &= \pi\int_{u=1}^{4} \tfrac{1}{2}u^{3/2} - \tfrac{1}{2}u^{1/2}\ du = \pi\Big[\tfrac{1}{5}u^{5/2} - \tfrac{1}{3}2^{3/2}\Big]_{u=1}^{4} = \pi\Big(\big(\tfrac{32}{5}-\tfrac{8}{3}\big) - \big(\tfrac{1}{5}-\tfrac{1}{3}\big)\Big) = \pi\big(\tfrac{31}{5}-\tfrac{7}{3}\big) = \tfrac{58\pi}{15}. \end{aligned}$$

5: Let F be a force field given by $F(x,y,z) = \big(x+y\,,\ x-y\,,\ z\big)$ and let C be the curve given parametrically by $(x,y,z) = \alpha(t) = \big(3t^2+1\,,\ 3t^2-1\,,\ 2t^3\big)$ for $0 \le t \le 1$.

(a) Find the length of the curve C.

Solution: We have $\alpha'(t) = (6t, 6t, 6t^2) = 6t(1,1,t)$ so that $|\alpha'(t)| = 6t\sqrt{2+t^2}$, and so the length is

$$L = \int_{t=0}^{1} |\alpha'(t)|\,dt = \int_{t=0}^{1} 6t\sqrt{t^2+1}\,dt = \Big[2(t^2+2)^{3/2}\Big]_{t=0}^{1} = 6\sqrt{3} - 4\sqrt{2}).$$

(b) Find work done by the force F acting on a small object which moves along C.

Solution: We have $F\big(\alpha(t)\big) = F(3t^2+1, 3t^2-1, 2t^3) = (6t^2, 2, 2t^3)$, and so the work is

$$W = \int_C F\cdot T\ dL = \int_{t=0}^{1} (6t^2,2,2t^3)\cdot(6t,6t,6t^2)\ dt = \int_{t=0}^{1} 36t^3 + 12t + 12t^5\ dt = \tfrac{36}{4} + \tfrac{12}{2} + \tfrac{12}{6} = 17.$$

6: A fluid flows in space with velocity field $V(x,y,z) = (yz, xz, z^2)$. Find the rate (volume per unit time) at which the fluid flows upwards through the surface

$$S = \Big\{(x,y,z)\,\Big|\,(x-1)^2+y^2 \le 1\,,\ z = \sqrt{x^2+y^2}\,\Big\}.$$

The surface S can be given parametrically by $(x,y,z) = \sigma(r,\theta) = \big(r\cos\theta\,,\ r\sin\theta\,,\ r\big)$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 2\cos\theta$.

Solution: We have

$$\begin{aligned} V\big(\sigma(r,\theta)\big) &= \big(r^2\sin\theta\,,\ r^2\cos\theta\,,\ r^2\big)\ \text{, and} \\ \sigma_r \times \sigma_\theta &= \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{pmatrix} \end{aligned}$$

and so the flux (which is the required rate of flow) is

$$\begin{aligned} \Phi &= \iint_S F\cdot N\ dA = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} \big(r^2\sin\theta, r^2\cos\theta, r^2\big)\cdot\big(-r\cos\theta\,,\ -r\sin\theta\,,\ r\big)\ dr\,d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} -2r^3\sin\theta + r^3\ dr\,d\theta = \int_{\theta=-\pi/2}^{\pi/2} (1-2\sin\theta)\Big[\tfrac{1}{4}r^4\Big]_{r=0}^{2\cos\theta}\ d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} (1-2\sin\theta\cos\theta)\cdot 4\cos^4\theta\ d\theta\,. \end{aligned}$$

Since $\sin\theta\cos^5\theta$ is an odd function and $\cos^4\theta$ is an even function, we have

$$\Phi = \int_{\theta=0}^{\pi/2} 8\cos^4\theta\ d\theta = \int_{\theta=0}^{\pi/2} 2(1+\cos 2\theta)^2\ d\theta = \int_{\theta=0}^{\pi/2} 2 + 4\cos 2\theta + 2\cos^2 2\theta\ d\theta = \pi + 0 + \tfrac{\pi}{2} = \tfrac{3\pi}{2}.$$

7: Find the flux of $F(x,y,z)=\left(x^2+\sqrt{z}\,,\,y^2+\sqrt{x}\,,\,3+x\right)$ outwards through the surface

$$S=\left\{(x,y,z)\middle|x^2+y^2+(z-1)^2=4\,,\,z\geq 0\right\}.$$

Solution: Let $D=\left\{(x,y,z)\middle|x^2+y^2+(z-1)^2\leq 4\,,\,z\geq 0\right\}$. The boundary surface of D consists of the given surface S together with the flat disc $T=\left\{(x,y,z)\middle|x^2+y^2\leq 3\,,\,z=0\right\}$. By the Divergence Theorem,

$$\iint_S F\bullet N\,dA+\iint_T F\bullet N\,dA=\iiint_D \nabla\bullet F\,dV$$

using the outwards normal vector for S and the downwards normal vector for T. We have $\nabla\bullet F=2x+2y$ and, by symmetry, we have $\iiint_D x\,dV=\iiint_D y\,dV=0$ and so $\iiint_D \nabla\bullet F\,dV=0$. Since $\iint_T x\,dA=0$ by symmetry, the outwards flux across S is

$$\begin{aligned}\iint_S F\bullet N\,dA&=-\iint_T F\bullet N\,dA=-\iint_T\left(x^2+\sqrt{7}\,,\,y^2+\sqrt{x}\,,\,3+x\right)\bullet(0,0,-1)\,dA\\&=\iint_T 3+x\,dA=3\iint_T 1\,dA=3\,\text{Area}(T)=3(3\pi)=9\pi.\end{aligned}$$

8: The square $S=\left\{(x,y,z)\middle|-1\leq x\leq 1\,,-1\leq y\leq 1\,,\,z=0\right\}$ carries a charge distribution with charge density (charge per unit area) $\rho(x,y)=|xy|$. Find the electric field at all points along the z-axis.

Solution: For $r=(0,0,z)$ and $s=(x,y,0)$ we have

$$dE(r)=\frac{1}{4\pi\epsilon_0}\cdot\frac{\rho(r-s)}{|r-s|^3}\,dA=\frac{1}{4\pi\epsilon_0}\cdot\frac{|xy|(-x,-y,z)}{(x^2+y^2+z^2)^{3/2}}\,dx\,dy.$$

By symmetry, the electric field at $r=(0,0,z)$ will be in the direction of the z-axis (pointing upwards for $z>0$ and downwards when $z<0$). Also the total electric field $E(r)$ is equal to 4 times the electric field produced by the portion of the square which lies in the first quadrant. The z-component of $E(r)$ is given by

$$\begin{aligned}E_z&=\int_{x=-1}^{1}\int_{y=-1}^{1}\frac{1}{4\pi\epsilon_0}\cdot\frac{|xy|\,z}{(x^2+y^2+z^2)^{3/2}}\,dy\,dx=4\int_{x=0}^{1}\int_{y=0}^{1}\frac{1}{4\pi\epsilon_0}\cdot\frac{xyz}{(x^2+y^2+z^2)^{3/2}}\,dy\,dx\\&=\frac{z}{\pi\epsilon_0}\int_{x=0}^{1}\left[\frac{-x}{(x^2+y^2+z^2)^{1/2}}\right]_{y=0}^{1}dx=\frac{z}{\pi\epsilon_0}\int_{x=0}^{1}\frac{x}{\sqrt{x^2+z^2}}-\frac{x}{\sqrt{x^2+z^2+1}}\,dx\\&=\frac{z}{\pi\epsilon_0}\left[\sqrt{x^2+z^2}-\sqrt{x^2+z^2+1}\right]_{x=0}^{1}=\frac{z}{\pi\epsilon_0}\Big(\big(\sqrt{z^2+1}-\sqrt{z^2+2}\big)-\big(\sqrt{z^2}-\sqrt{z^2+1}\big)\Big)\\&=\frac{1}{\pi\epsilon_0}\,z\Big(2\sqrt{z^2+1}-\sqrt{z^2+2}-|z|\Big).\end{aligned}$$

9: A loop of wire follows the line segment $L = \{(x,y,z)\big| -1 \le x \le 1\,,\, y = z = 0\}$ and the semicircle $C = \{(x,y,z)\big| x^2+y^2 = 1\,,\, y \ge 0\,,\, z = 0\}$, and it carries a constant current I. Find the magnetic field at all points along the z-axis.

Solution: First we find the contribution to the magnetic field made by the straight portion of the wire. For $r = (0,0,z)$ and $s = \alpha(t) = (t,0,0)$ for $-1 \le t \le 1$ we have

$$dB(r) = \frac{\mu_0}{4\pi}\cdot\frac{dI \times (r-s)}{|r-s|^3} = \frac{\mu_0 I}{4\pi}\frac{\alpha'(t)\times(r-s)}{|r-s|^3}\,dt = \frac{\mu_0}{4\pi}\cdot\frac{(1,0,0)\times(-t,0,z)}{(t^2+z^2)^{3/2}}\,dt = \frac{\mu_0 I}{4\pi}\cdot\frac{(0,-z,0)}{(t^2+z^2)^{3/2}}\,dt.$$

Making the substitution $z\tan\theta = dt$ so that $z\sec\theta = \sqrt{t^2+z^2}$ and $z\sec^2\theta\,d\theta = dt$, the y-component of this contribution is

$$\begin{aligned} B_y(r) &= \int_{t=-1}^{1}\frac{\mu_0}{4\pi}\cdot\frac{-z}{(t^2+z^2)^{3/2}}\,dt = -\frac{\mu_0 I}{4\pi}\int_{t=-1}^{1}\frac{z\cdot z\sec^2\theta\,d\theta}{(z\sec\theta)^{3/2}} = -\frac{\mu_0 I}{4\pi}\int_{t=-1}^{1}\frac{1}{z}\cos\theta\,d\theta \\ &= -\frac{\mu_0 I}{4\pi z}\Big[\sin\theta\Big]_{t=-1}^{1} = -\frac{\mu_0 I}{4\pi z}\left[\frac{t}{\sqrt{t^2+z^2}}\right]_{t=-1}^{1} = -\frac{\mu_0 I}{2\pi z}\,\frac{1}{\sqrt{1+z^2}}\,. \end{aligned}$$

Next we find the contribution made by the semicircular portion of the wire. For $r = (0,0,z)$ and $s = \alpha(t) = (\cos t, \sin t, 0)$ for $0 \le t \le \pi$ we have

$$dB(r) = \frac{\mu_0 I}{4\pi}\cdot\frac{\alpha'(t)\times(r-s)}{|r-s|^3} = \frac{\mu_0 I}{4\pi}\cdot\frac{(-\sin t,\cos t,0)\times(-\cos t,\sin t,z)}{(\cos^2 t+\sin^2 t+z^2)^{3/2}}\,dt = \frac{\mu_0 I}{4\pi}\cdot\frac{(z\cos t, z\sin t, 1)}{(1+z^2)^{3/2}}\,dt.$$

We calculate each component of this contribution: we have

$$\begin{aligned} B_x(r) &= \frac{\mu_0 I}{4\pi}\cdot\frac{z}{1+z^2}\int_0^\pi \cos t\,dt = 0\,, \\ B_y(r) &= \frac{\mu_0 I}{4\pi}\cdot\frac{z}{1+z^2}\int_0^\pi \sin t\,dt = \frac{\mu_0 I}{2\pi}\cdot\frac{z}{1+z^2}\,, \\ B_z(r) &= \frac{\mu_0 I}{4\pi}\cdot\frac{1}{1+z^2}\int_{t=0}^\pi 1\,dt = \frac{\mu_0 I}{4}\cdot\frac{1}{1+z^2}\,. \end{aligned}$$

Thus the total magnetic field induced by the loop of wire is

$$B(r) = \left(0\,,\,\frac{\mu_0 I}{2\pi}\Big(\frac{z}{1+z^2} - \frac{1}{z\sqrt{1+z^2}}\Big)\,,\,\frac{\mu_0 I}{4}\cdot\frac{1}{1+z^2}\right) = \frac{\mu_0 I}{4\pi z(1+z^2)}\Big(0\,,\,2\big(z^2-\sqrt{1+z^2}\big)\,,\,\pi z\Big).$$

10: A static charge distribution in space produces an electric field given by

$$E(x,y,z) = \sqrt{x^2+y^2+z^2}\,(x,y,z).$$

(a) Find the charge density $\rho = \rho(x,y,z)$.

Solution: We have

$$\tfrac{\partial}{\partial x}\big(x\sqrt{x^2+y^2+z^2}\,\big) = \sqrt{x^2+y^2+z^2} + x\cdot\tfrac{x}{\sqrt{x^2+y^2+z^2}} = \tfrac{2x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}$$

and similarly $\frac{\partial}{\partial y}\big(y\sqrt{x^2+y^2+z^2}\big) = \frac{x^2+2y^2+z^2}{\sqrt{x^2+y^2+z^2}}$ and $\frac{\partial}{\partial z}\big(z\sqrt{x^2+y^2+z^2}\big) = \frac{x^2+y^2+2z^2}{\sqrt{x^2+y^2+z^2}}$ so, by Maxwell's first equations (or Gauss' Law), the density is

$$\rho = \epsilon_0\nabla\cdot E = \epsilon_0\Big(\tfrac{2x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} + \tfrac{x^2+2y^2+z^2}{\sqrt{x^2+y^2+z^2}} + \tfrac{x^2+y^2+2z^2}{\sqrt{x^2+y^2+z^2}}\Big) = \tfrac{4\epsilon_0(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} = 4\epsilon_0\sqrt{x^2+y^2+z^2}.$$

(b) Find a scalar potential for E.

Solution: By inspection, if we let $g(x) = \frac{1}{3}(x^2+y^2+z^2)^{3/2}$ then we have $\nabla g = F$.

(c) Find the work done by E on an object of unit charge which moves along the curve given parametrically by $(x,y,z) = \alpha(t) = \big(t\,,\,t+1\,,\,t^2+t\big)$ for $1 \le t \le 2$.

Solution: By the Conservative Fields Theorem, the work is

$$\begin{aligned} W &= \int_\alpha F\cdot T\,dL = \int_\alpha(\nabla g)\cdot T\,dL = g\big(\alpha(2)\big) - g\big(\alpha(1)\big) \\ &= g(2,3,6) - g(1,2,2) = \tfrac{1}{3}\,49^{3/2} - \tfrac{1}{3}9^{3/2} = \tfrac{343-27}{3} = \tfrac{356}{3}. \end{aligned}$$

11: A steady current distribution in space produces a magnetic field given by

$$B(x,y,z) = \big(2z - 2y\,,\, 2x - z\,,\, y - 2x\big).$$

(a) Find the current density $J = J(x,y,z)$.

Solution: By Maxwell's fourth equation, the current density is

$$J = \tfrac{1}{\mu_0}\,\nabla\times B = \tfrac{1}{\mu_0}\big(1+1, 2+2, 2+2\big) = \tfrac{1}{\mu_0}\,(3,4,4).$$

(b) Find a vector potential for B.

Solution: By inspection, if we let $A = \big(2xy + 2xz\,,\, xy + 2yz\,,\, xz + 2yz\big)$ then we have $\nabla\times A = B$.

(c) Find the flux of B upwards through $S = \big\{(x,y,z)\big| x^2 + (y-1)^2 + z^2 = 1\,,\, z\geq 0\big\}$.

Solution: Let D be the half-ball $D = \big\{(x,y,z)\big| x^2 + (y-1)^2 + z^2 \leq 1\,,\, z \geq 0\big\}$. The boundary surface of D consists of the given surface S together with the flat disc $T = \big\{(x,y,z)\big| x^2 + (y-1)^2 \leq 1\,,\, z = 0\big\}$. Since $\nabla\cdot B = 0$, by the Divergence Theorem we have

$$\iint_S B\cdot N\ dA + \iint_T B\cdot N\ dA = \iiint_D \nabla\cdot B\ dV = 0$$

where we use the upwards flux through S and the downwards flux through T. Thus the required flux is

$$\begin{aligned}\Phi &= \iint_S B\cdot N\ dA = -\iint B\cdot N\ dA = -\iint_T \big(2z - 2y\,,\, 2x - z\,,\, y - 2x\big)\cdot(0,0,-1)\ dA \\ &= \iint_T y - 2x\ dA = \iint_T y\ dA = \tfrac{1}{2}\,\text{Volume}(C) = \tfrac{1}{2}(2\pi) = \pi,\end{aligned}$$

where we noticed that $\iint_T x\ dA = 0$ by symmetry, and then we found the value of $\iint_T y\ dA$ by inspection by noticing that this integral measures the volume of the solid given by $(x,y)\in T$, $0 \leq z \leq y$, which is equal half of the volume of the cylinder $C = \big\{(x,y,z)\big| x^2 + (y-1)^2 \leq 1\,,\, 0 \leq z \leq 1\big\}$.

12: (a) Show that a steady current density $J = J(x,y,z)$ must satisfy the requirement that whenever S is the boundary surface of a bounded region in $\mathbf{R}^3$ we have $\iint_S J\cdot N\ dA = 0$.

Solution: Let S be the boundary surface of the region D. Then by the Divergence Theorem and Maxwell's fourth equation we have

$$\iint_S J\cdot N\ dA = \iiint_D \nabla\cdot J\ dV = \iiint_D \nabla\cdot\big(\nabla\times \tfrac{1}{\mu_0}B\big)\ dV = \tfrac{1}{\mu_0}\iiint_D \nabla\cdot(\nabla\times B)\ dV = 0$$

since we have $\nabla\cdot(\nabla\times B) = 0$ (indeed $\nabla\cdot(\nabla\times F) = 0$ for every smooth vector field F).

(b) Given $k > 0$, find $\omega > 0$ such that the fields E and B given by

$$\begin{aligned}E(x,y,z,t) &= \big(\omega\sin(kz - \omega t)\,,\, 0\,,\, 0\big)\\ B(x,y,z,t) &= \big(0\,,\, k\sin(kz - \omega t)\,,\, 0\big)\end{aligned}$$

are solutions to Maxwell's Equations in a vacuum.

Solution: For the given fields E and B we have

$$\begin{aligned}\nabla\cdot E &= 0 & \nabla\cdot B &= 0\\ \nabla\times E &= \big(0\,,\, k\omega\,\cos(kz - \omega t)\,,\, 0\big) & \nabla\times B &= \big(-k^2\cos(kz - \omega t)\,,\, 0-\,,\, 0\big)\\ \tfrac{\partial E}{\partial t} &= \big(-\omega^2\cos(kz - \omega t)\,,\, 0\,,\, 0\big) & \tfrac{\partial B}{\partial t} &= \big(0\,,\, -k\omega\,\cos(kz - \omega t)\,,\, 0\big)\end{aligned}$$

Thus the first three of Maxwell's equations, namely $\nabla\cdot E = 0$, $\nabla\times E = -\frac{\partial B}{\partial t}$ and $\nabla\cdot B = 0$, are all satisfied, and Maxwell's fourth equation $\nabla\times B = \epsilon_0\mu_0\frac{\partial E}{\partial t}$ is satisfied when $k^2 = \epsilon_0\mu_0\omega^2$, that is when $\omega = k\big/\sqrt{\epsilon_0\mu_0}$.

13: Solve each of the following for $z \in \mathbf{C}$. Express your solutions in Cartesian form.

(a) $z^6 + 8 = 0$

Solution: For $z = re^{i\theta}$ we have

$$\begin{aligned} z^6+8=0 &\iff (re^{i\theta})^6=-8 \iff r^6e^{i\,6\theta}=8\,e^{i\,\pi} \iff \big(r^6=8 \text{ and } 6\theta=\pi+2\pi\,k \text{ for some } k\in\mathbf{Z}\big) \\ &\iff \big(r=\sqrt{2} \text{ and } \theta=\tfrac{\pi}{6}+\tfrac{k\pi}{3} \text{ for some } k\in\{0,1,2,3,4,5\}\big) \\ &\iff z=re^{i\theta}\in\big\{\sqrt{2}\,e^{i\pi/6},\ \sqrt{2}\,e^{i\,\pi/2},\ \sqrt{2}\,e^{i\,5\pi/6},\ \sqrt{2}\,e^{i\,7\pi/6},\ \sqrt{2}\,e^{i\,3\pi/2},\ \sqrt{2}\,e^{i\,11\pi/6}\big\}. \end{aligned}$$

In Cartesian form, the 6 solutions are $z \in \big\{ \pm\frac{\sqrt{3}}{2} \pm \frac{1}{\sqrt{2}}\,i\,,\ \pm\sqrt{2}\,i\big\}$.

(b) $iz^2 + (2+i)z + (7+i) = 0$

Solution: Using the Quadratic Formula, and noting that $(4-3i)^2 = (7-24i)$, the solutions are given by

$$z = \frac{-(2+i)\pm\sqrt{(3+4i)-4i(7+i)}}{2i} = \frac{-(2+i)\pm\sqrt{7-24i}}{2i} = \frac{-(2+i)\pm(4-3i)}{2i}$$

that is $z \in \big\{\frac{2-4i}{2i}, \frac{-6+2i}{2i}\big\} = \big\{-i(1-2i), -i(-3+i)\big\} = \big\{-2-i\,,\ 1+3i\big\}$.

(c) $z^3 + 6z + 2 = 0$ (hint: let $z = w - \frac{2}{w}$).

Solution: Let $z = w - \frac{2}{w}$. Then

$$\begin{aligned} z^3+6z+2=0 &\iff \big(w-\tfrac{2}{w}\big)^3+6\big(w-\tfrac{2}{w}\big)+2=0 \iff w^3-\tfrac{8}{w^3}+2=0 \\ &\iff w^6+2w^3-8=0 \iff (w^3+4)(w^3-2)=0 \iff w^3=2 \text{ or } w^3=-4 . \end{aligned}$$

Consider the case that $w^3 = 2$. Then $w \in \big\{\sqrt[3]{2},\ \sqrt[3]{2}\,e^{i\,2\pi/3},\ \sqrt[3]{2}\,e^{-i\,2\pi/3}\big\}$ and so

$$\begin{aligned} z=w-\tfrac{2}{w} &\in \Big\{\sqrt[3]{2}-\sqrt[3]{4},\ \sqrt[3]{2}\,e^{i\,2\pi/3}-\sqrt[3]{4}\,e^{-i\,2\pi/3},\ \sqrt[3]{2}\,e^{-i\,2\pi/3}-\sqrt[3]{4}\,e^{i\,2\pi/3}\Big\} \\ &= \Big\{\sqrt[3]{2}-\sqrt[3]{4},\ \sqrt[3]{2}\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\,i\right)-\sqrt[3]{4}\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}\,i\right),\ \sqrt[3]{2}\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}\,i\right)-\sqrt[3]{4}\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\,i\right)\Big\} \\ &= \Big\{-(\sqrt[3]{4}-\sqrt[3]{2}),\ \tfrac{1}{2}(\sqrt[3]{4}-\sqrt[3]{2})\pm\tfrac{\sqrt{3}}{2}(\sqrt[3]{4}-\sqrt[3]{2})\,i\Big\} \end{aligned}$$

Since a cubic polynomial has at most 3 roots, we have found all 3 solutions (so we do not need to consider the case that $w^3 = -4$).

14: (a) Use the formula $\tanh z = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$ to show that $\tanh^{-1} z = \frac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$.

Solution: We have

$$\begin{aligned} z=\tanh w &\iff z=\frac{e^w-e^{-w}}{e^w+e^{-w}} \iff ze^w+ze^{-w}=e^w-e^{-w} \iff ze^{2w}+z=e^{2w}-1 \\ &\iff z+1=(1-z)e^{2w} \iff e^{2w}=\tfrac{1+z}{1-z} \iff 2w=\log\tfrac{1+z}{1-z} \iff w=\tfrac{1}{2}\log\tfrac{1+z}{1-z}. \end{aligned}$$

(b) Solve $\tanh z = e^{i\,\pi/3}$.

Solution: By Part (a), we have $\tanh z = e^{i\,\pi/3}$ when

$$\begin{aligned} z &= \tanh^{-1}\left(e^{i\,\pi/3}\right)=\tanh^{-1}\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\,i\right)=\tfrac{1}{2}\log\frac{1+\frac{1}{2}+\frac{\sqrt{3}}{2}\,i}{1-\frac{1}{2}-\frac{\sqrt{3}}{2}\,i}=\tfrac{1}{2}\log\frac{3+\sqrt{3}\,i}{1-\sqrt{3}\,i}=\tfrac{1}{2}\log\frac{(3+\sqrt{3}\,i)(1+\sqrt{3}\,i)}{4} \\ &= \tfrac{1}{2}\log\frac{4\sqrt{3}\,i}{4}=\tfrac{1}{2}\log\left(\sqrt{3}e^{i\,\pi/2}\right)=\tfrac{1}{2}\Big(\ln\sqrt{3}+i\big(\tfrac{\pi}{2}+2\pi k\big)\Big)=\tfrac{1}{4}\ln 3+i\big(\tfrac{\pi}{4}+\pi k\big) \text{ for some } k\in\mathbf{Z}. \end{aligned}$$

(c) Solve $\sinh z = \dfrac{e^z}{1+i}$.

Solution: We have

$$\begin{aligned} \sinh z=\frac{e^z}{1+i} &\iff \frac{e^z-e^{-z}}{2}=\frac{e^z}{1+i} \iff (1+i)\big(e^z-e^{-z}\big)=2\,e^z \iff (-1+i)e^z=(1+i)e^{-z} \\ &\iff e^{2z}=\tfrac{1+i}{-1+i}=\tfrac{-2i}{2}=-i=e^{-i\,\pi/2} \iff 2z=\log(-i)=\log\left(e^{-i\,\pi/2}\right)=i\big(-\tfrac{\pi}{2}+2\pi k\big) \\ &\iff z=i\big(-\tfrac{\pi}{4}+k\pi\big) \text{ for some } k\in\mathbf{Z}. \end{aligned}$$

15: (a) Find the image under $w = f(z) = z^2$ of the line $x = c$ where $c > 0$.

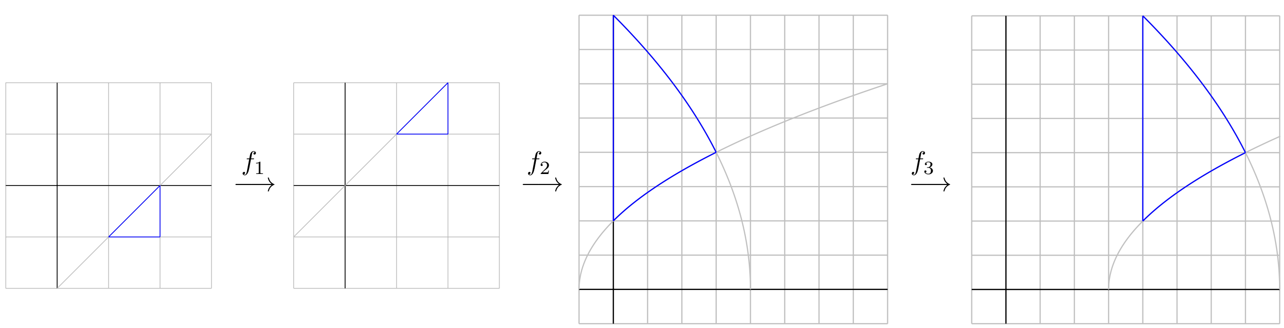
Solution: The line $x = c$ is given by $z = c+it$. It is mapped to the curve $w = z^2 = (c+it)^2 = (c^2-t^2)+i(2ct)$, that is the curve given by $u = c^2 - t^2$ (1) and $v = 2ct$ (2). From (2) we get $t = \frac{v}{2c}$ then from (1) we get $u = c^2 - \left(\frac{v}{2c}\right)^2$. Thus the image is the parabola $u = c^2 - \frac{1}{4c^2}\,v^2$, that is the parabola with vertex on the positive real axis at $w = c^2$ which opens to the left passing through the points $w = \pm i\,2c^2$.

(b) Show that $f(z) = z^2$ is equal to the composite $f = h \circ f \circ g$ where $g(z) = e^{-i\theta}z$ and $h(z) = e^{i2\theta}z$, and deduce that the image under $w = z^2$ of the line whose nearest point to the origin is the point $a = re^{i\theta}$ is equal to the parabola with vertex at $a^2 = r^2e^{i2\theta}$ which passes through the points $\pm i\,2a^2$.

Solution: Since $h\big(f\big(g(z)\big)\big) = h\big(f(e^{-i\theta}z)\big) = h\big((e^{-i\theta}z)^2\big) = h\big(e^{-i2\theta}z^2\big) = e^{i2\theta}e^{-i2\theta}z^2 = z^2 = f(z)$, it follows that $f = h \circ f \circ g$. The line whose nearest point to 0 is the point $a = re^{i\theta}$ is sent by the map g (which rotates clockwise by θ) to the line $x = r$. By Part (a), the map $x = r$ is then sent by the map f to the parabola with vertex at r^2 which passes through $\pm i\,2r^2$. This parabola is then sent by the map h (which rotates counterclockwise by 2θ) to the parabola with vertex at $r^2e^{i2\theta} = a^2$ which passes through the points $\pm i\,2r^2e^{i2\theta} = \pm i\,2a^2$, as required.

(c) Find the image under $f(z) = z^2 + 4i\,z$ of the triangle with vertices at 2, $1-i$ and $2-i$.

Solution: Let $a = 2$, $b = 1-i$ and $c = 2-i$. Note that $f(z) = z^2 + 4i\,z = (z+2i)^2 + 4 = f_3\big(f_2\big(f_1(z)\big)\big)$ where $f_1(z) = z + 2i$, $f_2(z_1) = {z_1}^2$ and $f_3(z_2) = z_2 + 4$. The first map $f_1(z) = z + 2i$ translates the given triangle up 2 units to the triangle with vertices at $a_1 = 2+2i$, $b_1 = 1+i$ and $c_1 = 2+i$. The second map $f_2(z_1) = {z_1}^2$ sends the three vertices to $a_2 = {a_1}^2 = 8i$, $b_2 = {b_1}^2 = 2i$ and $c_2 = {c_1}^2 = 3+4i$. The map $f_2(z) = z^2$ sends the ray from the origin in the direction of $e^{i\theta}$ to the ray through the origin in the direction of $e^{i\,2\theta}$, and it sends the line whose nearest point to the origin is the point $0 \neq u \in \mathbf{C}$ to the parabola with vertex at the point u^2 which passes through the points u^2 and $\pm 2i\,u^2$. Since the line segment a_1b_1 lies on the ray through the origin in the direction of $e^{i\pi/4}$, it is mapped to the line segment a_2b_2 which lies along the positive imaginary axis. The line segment b_1c_1 lies along the line whose nearest point to the origin is i. This line is mapped by f_2 to the parabola with vertex at $i^2 = -1$ which passes through the points -1 and $\pm 2i$. Thus the line segment b_1c_1 is mapped by f_2 to the arc along this parabola from $b_2 = 2i$ to $c_2 = 3+4i$. Similarly, the line segment c_1a_1 lies on the line whose nearest point to the origin is 2, and it is mapped by f_2 to the arc along the parabola with vertex at 4 from the point $b_2 = 3+4i$ to the point $a_2 = 8i$. Finally, the map f_3 translates the image 4 units to the right. The images are shown below.



16: Let $f\big(r\,e^{i\,\theta}\big) = r^{2/3}e^{i\,2\theta/3}$ for $r > 0$ and $0 < \theta < 2\pi$. Find $f'(-2+2i)$ and $f''(-2+2i)$. Express your answers in Cartesian form.

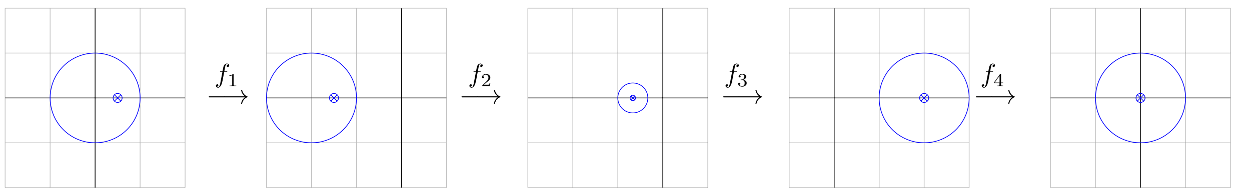
Solution: Note that $f(z)^3 = z^2$. Differentiate both sides, using the Chain Rule on the left side, to get $3f(z)^2f'(z) = 2z$ and so $f'(z) = \dfrac{2z}{3f(z)^2}$ and $f''(z) = \dfrac{2}{3}\cdot\dfrac{f(z)^2 - 2z\,f(z)f'(z)}{f(z)^4} = \dfrac{2}{3}\cdot\dfrac{f(z) - 2z\,f'(z)}{f(z)^3}$. When $z = -2+2i = 2\sqrt{2}\,e^{i\,3\pi/4}$ we have $f(z) = 2\,e^{i\,\pi/2} = 2i$ and so

$$f'(z) = \frac{2z}{3f(z)^2} = \frac{2(-2+2i)}{3(2i)^2} = \frac{1-i}{3}\ , \text{ and}$$

$$f''(z) = \frac{2}{3}\cdot\frac{f(z) - 2z\,f'(z)}{f(z)^3} = \frac{2}{3}\cdot\frac{(2i) - 2(-2+2i)\big(\frac{1-i}{3}\big)}{(2i)^3} = \frac{2}{3}\cdot\frac{6i - 2(4i)}{3(-8i)} = \frac{1}{36}.$$

17: Find the image under the map $f(z) = \dfrac{2z-1}{2-z}$ of the set $U = \left\{ z \in \mathbf{C} \middle| |z| < 1\,,\, z \neq \frac{1}{2} \right\}$.

Solution: Note that $f(z) = \dfrac{2z-1}{2-z} = \dfrac{2z-4+3}{2-z} = -2 - \dfrac{3}{z-2} = f_4\big(f_3\big(f_2(f_1(z))\big)\big)$ where $f_1(z) = z-2$, $f_2(z_1) = \frac{1}{z_1}$, $f_3(z_2) = -3z_2$ and $f_4(z_3) = z_3 - 2$. The map f_1 translates 2 units to the left and sends U to the set $U_1 = \left\{ z_1 \middle| |z_1+2| < 1\,,\, z_1 \neq -\frac{3}{2} \right\}$. Recall from a homework problem (Assignment 9, Problem 6) that the map $f_2(z_1) = \frac{1}{z_1}$ sends the circle with diameter a, ta to the circle with diameter $\frac{1}{a}, \frac{1}{ta}$, so it sends the circle with diameter $-1, -3$ to the circle with diameter $-1, -\frac{1}{3}$ (which is centered at $-\frac{2}{3}$ and has radius $\frac{1}{3}$, and hence it sends the set U_1 to the set $U_2 = \left\{ z_2 \middle| |z_2 + \frac{2}{3}| < \frac{1}{3}\,,\, z_2 \neq -\frac{2}{3} \right\}$. The map $f_3(z_2) = -3z_2$ rotates about the origin by the angle π and scales by the factor 3, and so it sends U_2 to the set $U_3 = \left\{ z_2 \middle| |z_2 - 2| < 1\,,\, z_2 \neq 2 \right\}$. Finally, the map f_4 translates 2 units to the left sending U_3 to the set $V = \left\{ w \middle| |w| < 1\,,\, w \neq 0 \right\}$. The images are shown below.



18: (a) Find $\displaystyle\int_\alpha z(3z-4)\,dz$ where $\alpha(t) = t + i$ for $0 \leq t \leq 1$.

Solution: Using the Fundamental Theorem of Calculus we have

$$\begin{aligned}\int_\alpha f &= \int_\alpha 3z^2 - 4z\;dz = \Big[z^3 - 2z\Big]_{\alpha(0)}^{\alpha(1)} = \Big[z^3 - 2z^2\Big]_i^{1+i} = (1+i)^3 - 2(1+i)^2 - (i)^3 + 2(i)^2 \\ &= (1+i)(2i) - 2(2i) + i - 2 = -4 - i.\end{aligned}$$

(b) Find $\displaystyle\int_\alpha f(z)\,dz$ where $f\big(r\,e^{i\,\theta}\big) = r^{1/3}e^{i\,\theta/3}$ for $r > 0$ and $-\pi < \theta < \pi$ and $\alpha(t) = 2 + it$ for $-2 \leq t \leq 2$.

Solution: Since $f(z)$ is a branch of $z^{1/3}$, we expect that an antiderivative $g(z)$ is given by a branch of $\frac{3}{4}\,z^{4/3}$, but let us take some care in deciding exactly which branch to use. Let $U = \left\{ re^{i\theta} \middle| r > 0\,,\, -\frac{3\pi}{4} < \theta < \frac{3\pi}{4} \right\}$ and note that when $-\frac{3\pi}{4} < \theta < \frac{3\pi}{4}$ we have $-\pi < \frac{4\theta}{3} < \pi$. Define $g : U \to \mathbf{C}$ by

$$g\big(re^{i\theta}\big) = \tfrac{3}{4}\,r^{4/3}e^{i\,4\theta/3} \quad \text{for } r > 0 \text{ and } -\tfrac{3\pi}{4} < \theta < \tfrac{3\pi}{4}.$$

Note that $g(z)^3 = \frac{27}{64}z^4$ so that $3g(z)^2g'(z) = \frac{27}{16}z^3$, that is $g'(z) = 9z^3\big/16g(z)^2$. For $z = re^{i\theta}$ we have

$$g'(z) = \frac{9z^3}{16g(z)^2} = \frac{9r^3e^{i\,3\theta}}{16\big(\frac{3}{4}\,r^{4/3}e^{i\,4\theta/3}\big)^2} = \frac{r^3e^{i\,3\theta}}{r^{8/3}e^{i\,8\theta/3}} = r^{1/3}e^{i\,\theta/3} = f(z)$$

and so g is indeed an antiderivative of f in the set U. By the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_\alpha f &= \Big[g(z)\Big]_{\alpha(-2)}^{\alpha(2)} = \Big[g(z)\Big]_{2-2i}^{2+2i} = g(2+2i) - g(2-2i) = g\big(2\sqrt{2}\,e^{i\,\pi/4}\big) - g\big(2\sqrt{2}\,e^{-i\,\pi/4}\big) \\ &= \tfrac{3}{4}\cdot 4e^{i\pi/3} - \tfrac{3}{4}\cdot 4e^{-i\,\pi/3} = 3\big(e^{i\,\pi/3} - e^{-i\,\pi/3}\big) = 3\cdot 2i\,\sin\tfrac{\pi}{3} = 3\sqrt{3}\,i.\end{aligned}$$

19: Find $\displaystyle\int_\alpha \frac{4\,dz}{(z+1)^2(z^2+1)}$ where $\alpha(t)=1+t(-3+i)$ for $0\le t\le 1$.

Solution: To get $\dfrac{A}{z+1}+\dfrac{B}{(z+1)^2}+\dfrac{C}{z+i}+\dfrac{D}{z-i}=\dfrac{4}{(z+1)^2(z^2+1)}$ we need

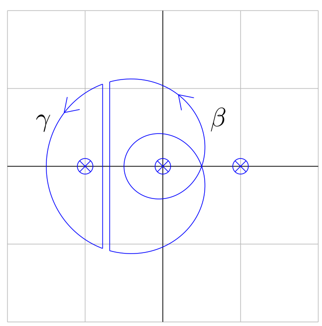
$$A(z+1)(z^2+1)+B(z^2+1)+C(z+1)^2(z-i)+D(z+1)^2(z+i)=4\,.$$

Putting in $z=-1$ gives $2B=4$ so $B=2$. Putting in $z=i$ gives $-4D=4$ so $D=-1$. Putting in $z=-i$ gives $-4C=4$ so $C=-1$. Equating coefficients of z^3 gives $A+C+D=0$ so $A=-C-D=2$. By the Fundamental Theorem of Calculus and the Winding Number Theorem (it helps to make a sketch of $\alpha(t)$, which is the line segment from $\alpha(0)=1$ to $\alpha(1)=-2+i$ to determine the values of $r(0)$, $r(1)$, $\theta(0)$ and $\theta(1)$ when using the Winding Number Theorem), we have

$$\begin{aligned}\int_\alpha \frac{4\,dz}{(z+1)^2(z^2+1)} &= \int_\alpha \frac{2}{z+1}+\frac{2}{(z+1)^2}-\frac{1}{z+i}-\frac{1}{z-i}\,dz\\ &=2\left(\ln\tfrac{\sqrt{2}}{2}+i\,\tfrac{3\pi}{4}\right)+2\left[\tfrac{-1}{z+1}\right]_1^{-2+i}-\left(\ln\tfrac{2\sqrt{2}}{\sqrt{2}}+i\,\tfrac{\pi}{2}\right)-\left(\ln\tfrac{2}{\sqrt{2}}-i\,\tfrac{3\pi}{4}\right)\\ &=\left(-\ln 2+i\,\tfrac{3\pi}{2}\right)+2\left(\tfrac{-1}{-1+i}+\tfrac12\right)-\left(\ln 2+i\,\tfrac{\pi}{2}\right)-\left(\tfrac12\ln 2-i\,\tfrac{3\pi}{4}\right)\\ &=-\tfrac52\ln 2+i\,\tfrac{7\pi}{4}+2\left(\tfrac{1+i}{2}+\tfrac12\right)=\left(2-\tfrac{5\ln 2}{2}\right)+i\left(1+\tfrac{7\pi}{4}\right)\,.\end{aligned}$$

20: Find $\displaystyle\int_\alpha \frac{\log(z-1)}{z(z+1)^3}\,dz$ where $\alpha(t)=\frac12+\frac12(1-3\cos t)e^{it}$ for $0\le t\le 2\pi$, as shown below, and $\log(w)$ is given by $\log\left(r\,e^{i\,\theta}\right)=\ln(r)+i\,\theta$ for $r>0$ and $0<\theta<2\pi$.

Solution: We decompose α, say along the line $\text{Re}(z)=-\frac34$, into two loops β and γ so that β winds twice around $z=0$ (but does not surround $z=-1$) and γ winds once around $z=-1$ (but does not surround $z=0$), as shown below.



By Cauchy's Integral Formulas we have

$$\int_\alpha \frac{\log(z-1)}{z(z+1)^3}\,dz=\int_\beta \frac{F(z)}{z}\,dz+\int_\gamma \frac{G(z)}{(z+1)^3}\,dz=2\pi i\cdot 2\cdot F(0)+2\pi i\cdot 1\cdot\frac{G''(-1)}{2!}$$

where $F(z)=\dfrac{\log(z-1)}{(z+1)^3}$ and $G(z)=\dfrac{\log(z-1)}{z}$. Note that we have $F(0)=\log(-1)=i\pi$ and we have $G'(z)=\dfrac{\frac{z}{z-1}-\log(z-1)}{z^2}=\dfrac{1}{z^2-z}-\dfrac{\log(z-1)}{z^2}$ and $G''(z)=\dfrac{-(2z-1)}{(z^2-z)^2}-\dfrac{\frac{z^2}{z-1}-2z\log(z-1)}{z^4}$ so that $G''(-1)=\frac34-\left(-\frac12+2\log(-2)\right)=\frac54-2(\ln 2+i\pi)$. Thus

$$\int_\alpha \frac{\log(z-1)}{z(z+1)^3}\,dz=4\pi i\,(i\pi)+\pi i\left(\tfrac54-2\ln 2-i\,2\pi\right)=-2\pi^2+i\,\pi\left(\tfrac54-2\ln 2\right).$$