ECE 206 Advanced Calculus 2, Review Problems

1: Find an implicit equation, of the form $ax+by+cz=d$, for the tangent plane to the surface given parametrically by $(x,y,z)=\sigma(s,t)=\big(\sqrt{s}\,,\,(2-s)\cos t\,,\,(2-s)\sin t\big)$ for $0\le s\le 5$ and $0\le t\le 2\pi$ at the point where $(s,t)=\big(4,\frac{\pi}{6}\big)$.

2: Let $(x,y)=f(r,\theta)=\big(r\cos\theta\,,\,r\sin\theta\big)$, and $z=g(x,y)$, and let $h(r,\theta)=g\big(f(r,\theta)\big)$. Suppose that $h(r,\theta)=r^2e^{\sqrt{3}\,(\theta-\frac{\pi}{6})}$. Use the Chain Rule to find $\nabla g(\sqrt{3},1)$.

3: Find the total charge in the solid tetrahedron with vertices at $(0,1,0)$, $(1,0,0)$, $(3,1,0)$ and $(3,1,3)$ where the charge density (charge per unit volume) is given by $\rho(x,y,z)=y$.

4: Find the total mass on the surface $S=\big\{(x,y,z)\big|x^2+y^2\le 3\,,\,2z=x^2+y^2\big\}$ with density (mass per unit area) $\rho(x,y,z)=x^2$.

5: Let F be a force field given by $F(x,y,z)=\big(x+y\,,\,x-y\,,\,z\big)$ and let C be the curve given parametrically by $(x,y,z)=\alpha(t)=\big(3t^2+1\,,\,3t^2-1\,,\,2t^3\big)$ for $0\le t\le 1$.

(a) Find the length of the curve C.

(b) Find work done by the force F acting on a small object which moves along C.

6: A fluid flows in space with velocity field $V(x,y,z)=(yz,xz,z^2)$. Find the rate (volume per unit time) at which the fluid flows upwards through the surface

$$S=\left\{(x,y,z)\Big|(x-1)^2+y^2\le 1\,,\,z=\sqrt{x^2+y^2}\right\}.$$

The surface S can be given parametrically by $(x,y,z)=\sigma(r,\theta)=\big(r\cos\theta\,,\,r\sin\theta\,,\,r\big)$ with $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ and $0\le r\le 2\cos\theta$.

7: Find the flux of $F(x,y,z)=\big(x^2+\sqrt{z}\,,\,y^2+\sqrt{x}\,,\,3+x\big)$ outwards through the surface $S=\big\{(x,y,z)\big|x^2+y^2+(z-1)^2=4\,,\,z\ge 0\big\}$.

8: The square $S=\big\{(x,y,z)\big|-1\le x\le 1\,,\,-1\le y\le 1\,,\,z=0\big\}$ carries a charge distribution with charge density (charge per unit area) $\rho(x,y)=|xy|$. Find the electric field at all points along the z-axis.

9: A loop of wire follows the line segment $L=\big\{(x,y,z)\big|-1\le x\le 1\,,\,y=z=0\big\}$ and the semicircle $C=\big\{(x,y,z)\big|x^2+y^2=1\,,\,y\ge 0\,,\,z=0\big\}$, and it carries a constant current I. Find the magnetic field at all points along the z-axis.

10: A static charge distribution in space produces an electric field given by

$$E(x,y,z)=\sqrt{x^2+y^2+z^2}\,(x,y,z).$$

(a) Find the charge density $\rho=\rho(x,y,z)$.

(b) Find a scalar potential for E.

(c) Find the work done by E on an object of unit charge which moves along the curve given parametrically by $(x,y,z)=\alpha(t)=\big(t\,,\,t+1\,,\,t^2+t\big)$ for $1\le t\le 2$.

11: A steady current distribution in space produces a magnetic field given by

$$B(x,y,z) = \big(2z - 2y\,,\ 2x - z\,,\ y - 2x\big).$$

(a) Find the current density $J = J(x,y,z)$.

(b) Find a vector potential for B.

(c) Find the flux of B upwards through $S = \big\{(x,y,z)\big|x^2 + (y-1)^2 + z^2 = 1\,,\ z \geq 0\big\}$.

12: (a) Show that a steady current density $J = J(x,y,z)$ must satisfy the requirement that whenever S is the boundary surface of a bounded region in $\mathbf{R}^3$ we have $\displaystyle\iint_S J \bullet N\, dA = 0$.

(b) Given $k > 0$, find $\omega > 0$ such that the fields E and B given by

$$E(x,y,z,t) = \big(\omega \sin(kz - \omega t)\,,\ 0\,,\ 0\big)$$
$$B(x,y,z,t) = \big(0\,,\ k \sin(kz - \omega t)\,,\ 0\big)$$

are solutions to Maxwell's Equations in a vacuum.

13: Solve each of the following for $z \in \mathbf{C}$. Express your solutions in Cartesian form.

(a) $z^6 + 8 = 0$

(b) $iz^2 + (2+i)z + (7+i) = 0$

(c) $z^3 + 6z + 2 = 0$ (hint: let $z = w - \frac{2}{w}$).

14: (a) Use the formula $\tanh z = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$ to show that $\tanh^{-1} z = \frac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$.

(b) Solve $\tanh z = e^{i\,\pi/3}$.

(c) Solve $\sinh z = \dfrac{e^z}{1+i}$.

15: (a) Find the image under $w = f(z) = z^2$ of the line $x = c$ where $c > 0$.

(b) Show that $f(z) = z^2$ is equal to the composite $f = h \circ f \circ g$ where $g(z) = e^{-i\theta}z$ and $h(z) = e^{i2\theta}z$, and deduce that the image under $w = z^2$ of the line whose nearest point to the origin is the point $a = re^{i\theta}$ is equal to the parabola with vertex at $a^2 = r^2e^{i2\theta}$ which passes through the points $\pm i\,2a^2$.

(c) Find the image under $f(z) = z^2 + 4i\,z$ of the triangle with vertices at 2, $1-i$ and $2-i$.

16: Let $f\big(r\,e^{i\,\theta}\big) = r^{2/3}e^{i\,2\theta/3}$ for $r > 0$ and $0 < \theta < 2\pi$. Find $f'(-2+2i)$ and $f''(-2+2i)$. Express your answers in Cartesian form.

17: Find the image under the map $f(z) = \dfrac{2z-1}{2-z}$ of the set $U = \big\{z \in \mathbf{C}\big||z| < 1\,,\ z \neq \frac{1}{2}\big\}$.

18: (a) Find $\displaystyle\int_\alpha z(3z-4)\,dz$ where $\alpha(t) = t + i$ for $0 \leq t \leq 1$.

(b) Find $\displaystyle\int_\alpha f(z)\,dz$ where $f\big(r\,e^{i\,\theta}\big) = r^{1/3}e^{i\,\theta/3}$ for $r > 0$ and $-\pi < \theta < \pi$ and $\alpha(t) = 2+it$ for $-2 \leq t \leq 2$.

19: Find $\displaystyle\int_\alpha \frac{4\,dz}{(z+1)^2(z^2+1)}$ where $\alpha(t)=1+t(-3+i)$ for $0\le t\le 1$.

20: Find $\displaystyle\int_\alpha \frac{\log(z-1)}{z(z+1)^3}\,dz$ where $\alpha(t)=\frac{1}{2}+\frac{1}{2}(1-3\cos t)e^{it}$ for $0\le t\le 2\pi$, as shown below, and $\log(w)$ is given by $\log\left(r\,e^{i\,\theta}\right)=\ln(r)+i\,\theta$ for $r>0$ and $0<\theta<2\pi$.