Lecture Notes on Vector Calculus

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Chapter 1. Functions

1.1 Definition: Let $D \subseteq \mathbf{R}^n$. We say that f is a **function** or a **map** from D to $\mathbf{R}^m$, and we write $f : D \subseteq \mathbf{R}^n \to \mathbf{R}^m$, when for every $x \in D$ there is a unique point $y = f(x) \in \mathbf{R}^m$. The set D is called the **domain** of the function f.

The **graph** of the function f is the set

$$\text{Graph}\,(f) = \big\{(x, f(x))\big| x \in D\big\} \subseteq \mathbf{R}^{n+m}.$$

We say the graph of f is defined **explicitly** by the equation $y = f(x)$.

The **null set** of f is the set

$$\text{Null}\,(f) = f^{-1}(0) = \big\{x \in D \big| f(x) = 0\big\} \subseteq \mathbf{R}^n.$$

More generally, given $k \in \mathbf{R}^m$, the **level set** $f^{-1}(k)$, also called the **inverse image** of k under f, is the set

$$f^{-1}(k) = \big\{x \in D \big| f(x) = k\big\} \subseteq \mathbf{R}^n.$$

More generally still, given a subset $B \subseteq \mathbf{R}^n$, the **inverse image** of B under f is the set

$$f^{-1}(B) = \big\{x \in D \big| f(x) \in B\big\} \subseteq \mathbf{R}^n.$$

We say the level set $f^{-1}(k)$ is defined **implicitly** by the equation $f(x) = k$.

The **range** of f, also called the **image** of f, is the set

$$\text{Range}\,(f) = f(D) = \big\{f(x))\big| x \in D\big\} \subseteq \mathbf{R}^m.$$

More generally, given a set $A \subseteq D$, the **image** of A under f is the set

$$f(A) = \big\{f(x)\big| x \in A\big\} \subseteq \mathbf{R}^m.$$

We say the range of f is defined **parametrically** by the equation $y = f(x)$, and for $x = (x_1, x_2, \cdots, x_n) \in D$, the variables $x_1, x_2, \cdots, x_n$ are called the **parameters**.

1.2 Note: The graph, the level sets and the range of a function $f : D \subseteq \mathbf{R}^n \to \mathbf{R}^m$ are geometric objects such as points, curves, surfaces, or higher dimensional analogues of these. In accordance with the above definitions, a curve in $\mathbf{R}^2$ or in $\mathbf{R}^3$, or a surface in $\mathbf{R}^3$, can be defined explicitly, implicitly, or parametrically.

A curve in $\mathbf{R}^2$ can be defined explicitly as the graph of a function $f : D \subseteq \mathbf{R} \to \mathbf{R}$, implicitly as the null set (or a level set) of a function $g : D \subseteq \mathbf{R}^2 \to \mathbf{R}$, or parametrically as the range of a function $\alpha : D \subseteq \mathbf{R} \to \mathbf{R}^3$.

A curve in $\mathbf{R}^3$ can be defined explicitly as the graph of a function $f : D \subseteq \mathbf{R} \to \mathbf{R}^2$, implicitly as the null set (or a level set) of a function $g : D \subseteq \mathbf{R}^3 \to \mathbf{R}^2$, or parametrically as the range of a map $\alpha : D \subseteq \mathbf{R} \to \mathbf{R}^3$.

A surface in $\mathbf{R}^3$ can be defined explicitly as the graph of a function $f : \mathbf{R}^2 \to \mathbf{R}$, implicitly as the null set (or as a level set) of a function $g : D \subseteq \mathbf{R}^3 \to \mathbf{R}$, or parametrically as the range of a function $\sigma : D \subseteq \mathbf{R}^2 \to \mathbf{R}^3$.

1.3 Example: Consider the unit circle $x^2+y^2=1$ in $\mathbf{R}^2$. For $f:[-1,1]\subseteq\mathbf{R}\to\mathbf{R}$ given by $f(x)=\sqrt{1-x^2}$, the graph of f, that is the curve $y=f(x)$, is equal to the top half of the unit circle. For $g:\mathbf{R}^2\to\mathbf{R}$ given by $g(x,y)=x^2+y^2-1$, the null set of g, that is the curve $x^2+y^2=1$, is equal to the entire circle. For $\alpha:\mathbf{R}\to\mathbf{R}$ given by $\alpha(t)=(\cos t,\sin t)$, the range of α, that is the curve $(x,y)=\alpha(t)$, is equal to the entire circle.

1.4 Example: Consider the ellipse which is the intersection of the cylinder $x^2+y^2=1$ with the plane $z=x+y$ in $\mathbf{R}^3$. The ellipse is given implicitly by the two equations $x^2+y^2=1$ and $z=x+y$, which can be written in vector form as the single equation $\big(x^2+y^2-1\,,\,z-x-y\big)=(0,0)$, and so it is the null set of the function $g:\mathbf{R}^3\to\mathbf{R}^2$ given by $g(x,y,z)=\big(x^2+y^2-1\,,\,z-x-y\big)$. To obtain a parametric description of the ellipse, note that to get $x^2+y^2=1$ we can take $x=\cos t$ and $y=\sin t$, and then to get $z=x+y$ we can take $z=\cos t+\sin t$, and so the ellipse is given parametrically by $(x,y,z)=\big(\cos t\,,\,\sin t\,,\,\cos t+\sin t\big)$. In other words, the ellipse is the range of the function $\alpha:\mathbf{R}\to\mathbf{R}^3$ given by $\alpha(t)=\big(\cos t\,,\,\sin t\,,\,\cos t+\sin t\big)$. To obtain an explicit description for half of the ellipse, note that the top half of the circle $x^2+y^2=1$ is given by $y=\sqrt{1-x^2}$ and then to get $z=x+y$ we need $z=x+\sqrt{1-x^2}$, and so the right half of the ellipse (when the y-axis points to the right) is given explicitly by $(y,z)=\big(\sqrt{1-x^2}\,,\,x+\sqrt{1-x^2}\big)$. In other words, the right half of the ellipse is the graph of the function $g:[-1,1]\subseteq\mathbf{R}\to\mathbf{R}^2$ given by $g(x)=\big(\sqrt{1-x^2}\,,\,x+\sqrt{1-x^2}\big)$.

1.5 Example: Consider the unit sphere in $\mathbf{R}^3$ given by $x^2+y^2+z^2=1$. The top half of the sphere is the graph $z=f(x,y)$ where $f:D\subseteq\mathbf{R}^2\to\mathbf{R}$ is given by $f(x,y)=\sqrt{1-x^2-y^2}$ with $D=\big\{(x,y)\in\mathbf{R}^2\big|x^2+y^2\leq 1\big\}$. The entire sphere is the null set $g(x,y,z)=0$ where $g:\mathbf{R}^3\to\mathbf{R}$ is given by $g(x,y,z)=x^2+y^2+z^2-1$. The top half of the sphere can be given parametrically by $x=r\cos\theta$ and $y=r\sin\theta$ and $z=\sqrt{1-r^2}$, so it is the range $(x,y,z)=\sigma(r,\theta)$ where $\sigma:D\subseteq\mathbf{R}^2\to\mathbf{R}^3$ is given by $\sigma(r,\theta)=\big(r\cos t\,,\,r\sin t\,,\,\sqrt{1-r^2}\big)$ with $D=\big\{(r,\theta)\in\mathbf{R}^2\big|0\leq r\leq 1\big\}$.

1.6 Remark: A function is uniquely determined by its graph but not by its null set or by its image. It follows that implicit and parametric descriptions of curves and surfaces are not unique. For example, the parabola $y=x^2$ can be given implicitly by $g(x,y)=0$ for any of the functions $g(x,y)=y-x^2$, $g(x,y)=(y-x^2)^3$ or $g(x,y)=(y-x^2)(x^2+1)$, and it can be given parametrically by $(x,y)=\alpha(t)$ for any of the functions $\alpha(t)=(t,t^2)$, $\alpha(t)=(t^3,t^6)$ or $\alpha(t)=(\sinh t\,,\,\sinh^2 t)$.

1.7 Remark: Given an explicit equation for a curve or surface it is easy to obtain an implicit or parametric equation for the curve or surface. For example the curve $y=f(x)$ in $\mathbf{R}^2$ can be given implicitly by $g(x,y)=0$ where $g(x,y)=y-f(x)$ and parametrically by $(x,y)=\alpha(t)$ where $\alpha(t)=\big(t,f(t)\big)$, Similarly the surface $z=f(x,y)$ in $\mathbf{R}^3$ can be given implicitly by $g(x,y,z)=0$ where $g(x,y,z)=z-f(x,y)$ and parametrically by $(x,y,z)=\sigma(s,t)$ where $\sigma(s,t)=\big(s,t,f(s,t)\big)$. On the other hand, given an implicit or a parametric equation for a curve or a surface it can be difficult or impossible to obtain an explicit equation.

1.8 Exercise: The **helix** is given explicitly by $x=\cos z$ and $y=\sin z$. Sketch the curve and find an implicit and a parametric equation for the curve.

1.9 Exercise: The **alpha curve** is given implicitly by $y^2=x^3+x$. Sketch the curve, find explicit equations for the top and bottom halves of the curve, and find a parametric equation for the entire curve.

1.10 Exercise: The curve which is given explicitly in polar coordinates by $r = r(\theta)$ is given parametrically in Cartesian coordinates by $(x,y) = \alpha(t) = \big(r(t)\cos t\,,\, r(t)\sin t\big)$. Sketch the **cardioid** which is given in polar coordinates by $r = r(\theta) = 1 + \cos\theta$, then find an implicit equation for the curve.

1.11 Exercise: The **twisted cubic** is given parametrically by $(x,y,z) = \alpha(t) = (t, t^2, t^3)$. Sketch the curve and find an implicit and an explicit equation for the curve.

1.12 Remark: In order to sketch a surface which is defined explicitly as a graph $z = f(x,y)$ or implicitly as a level set $g(x,y,z) = k$, it often helps to first sketch curves of intersection of the surface with various planes $x = c$, $y = c$ or $z = c$. The intersection of the graph $z = f(x,y)$ with the plane $z = c$ is given implicitly by $f(x,y) = c$. The intersection of the level set $g(x,y,z) = k$ with the plane $z = c$ is given implicitly by $g(x,y,c) = k$

1.13 Exercise: Sketch the curve of intersection of the cylinder $x^2 + y^2 = 1$ with the parabolic sheet $z = x^2$ and find implicit, explicit, and parametric equations for the curve.

1.14 Exercise: Sketch the surface $z = x^2 + y^2$.

1.15 Exercise: Sketch the surface $z = 4x^2 - y^2$.

1.16 Exercise: Sketch the surface $x^2 + 4y^2 - z^2 = 0$.

1.17 Exercise: Sketch the surface $(x,y,z) = \sigma(u,v) = \big(u, v, u^2 + 4v^2 - 3\big)$.

1.18 Exercise: Find a parametric equation $(x,y,z) = \sigma(\phi,\theta)$ for the sphere of radius r centred at the origin, where the parameters ϕ and θ are the angles of latitude and longitude. In other words, find $\sigma(\phi,\theta)$ so that when $(x,y,z) = \sigma(\phi,\theta)$, ϕ is the angle between $(0,0,1)$ and (x,y,z) and θ is the angle from $(1,0)$ counterclockwise to (x,y).

1.19 Exercise: Find implicit and parametric equations for the **torus** which is obtained by rotating the circle $(x,z) = \big(R + r\cos\theta\,,\, r\sin\theta\big)$ about the z-axis.

1.20 Definition: An **affine space** in $\mathbf{R}^n$ is a set of the form $p + V = \{p + v | v \in V\}$ for some $p \in \mathbf{R}^n$ and some vector space $V \subseteq \mathbf{R}^n$. The **dimension** of the affine space $p + V$ is the same as the dimension of V. The set $p + V$ is called the affine space through p parallel to V, or the affine space through p perpendicular to $V^\perp$ (the orthogonal complement of V).

1.21 Example: In $\mathbf{R}^3$, the only zero dimensional vector space is the origin $\{0\}$, the 1-dimensional vector spaces are the lines through the origin, the 2-dimensional spaces are the planes through the origin, and the only 3-dimensional vector space is all of $\mathbf{R}^3$. The 0-dimensional affine spaces are the points in $\mathbf{R}^3$, the 1-dimensional affine spaces are the lines in $\mathbf{R}^3$, the 2-dimensional affine spaces are the planes in $\mathbf{R}^3$, and the only 3-dimensional affine space is all of $\mathbf{R}^3$.

1.22 Definition: Let $f : \mathbf{R}^n \to \mathbf{R}^m$. The function f is called **linear** when it is of the form $f(x) = Ax$ for some matrix $A \in M_{m\times n}(\mathbf{R})$, and f is called **affine** when it is of the form $f(x) = Ax + b$ for some matrix $A \in M_{m\times n}(\mathbf{R})$ and some vector $b \in \mathbf{R}^m$.

1.23 Note: Let $A \in M_{m\times n}$ and let f be the linear map $f(x) = Ax$. Let $u_1, \cdots, u_n$ be the column vectors of A and let $v_1, \cdots, v_m$ be the row vectors of A so that we have $A = \big(u_1, \cdots, u_n\big) = \big(v_1, \cdots, v_m\big)^T$. Let c be a point in the range of f, say $f(p) = c$. Then

$$\begin{aligned}
\text{Range}\,(f) &= \big\{Ax\big|x \in \mathbf{R}^n\big\} = \Big\{\sum_{i=1}^{n} u_i x_i\Big| \text{ each } x_i \in \mathbf{R}\Big\} = \text{Span}\{u_1, \cdots, u_n\} = \text{Col}(A), \\
\text{Null}\,(f) &= \text{Null}\,(A) = \big\{x \in \mathbf{R}^n\big|Ax = 0\big\} = \big\{x \in \mathbf{R}^n\big|v_i \cdot x = 0 \text{ for all } i\big\} = \text{Row}(A)^\perp, \\
f^{-1}(c) &= \big\{x \in \mathbf{R}^n\big|Ax = c\big\} = \big\{x \in \mathbf{R}^n\big|Ax = Ap\big\} = \big\{x \in \mathbf{R}^n\big|A(x-p) = 0\big\} \\
&= \big\{p + y \in \mathbf{R}^n\big|Ay = 0\big\} = p + \text{Null}\,(A).
\end{aligned}$$

1.24 Note: Let $A \in M_{m\times n}(\mathbf{R})$, let $b \in \mathbf{R}^m$ and let $f(x) = Ax + b$. Let c be in the range of f with say $f(p) = c$. Then

$$\begin{aligned}
\text{Range}\,(f) &= \big\{Ax + b\big|x \in \mathbf{R}^n\big\} = b + \text{Col}(A)\ , \text{ and} \\
f^{-1}(c) &= \big\{x \in \mathbf{R}^n\big|Ax + b = c = Ap + b\big\} = \big\{x \in \mathbf{R}^n\big|A(x-p) = 0\big\} = p + \text{Null}\,(A).
\end{aligned}$$

Note that if $u_1, u_2, \cdots, u_n$ are the columns of A and $e_1, e_2, \cdots, e_n$ are the standard basis vectors for $\mathbf{R}^n$, then we have $f(0) = b$ and $f(e_i) = Ae_i + b = u_i + b$. If $v_1, \cdots, v_m$ are the row vectors of A and $k = c - b$, then since

$$f(x) = c \iff Ax + b = c \iff Ax = k \iff v_i \cdot x = k_i \text{ for all } i,$$

it follows that the level set $f(x) = c$ is the intersection of the affine spaces $v_i \cdot x = k_i$, and we note that the space $v_i \cdot x = k_i$ is the affine space in $\mathbf{R}^m$ of dimension $m - 1$ through p perpendicular to v_i.

1.25 Exercise: Define $f : \mathbf{R}^3 \to \mathbf{R}^2$ by $f(x, y, z) = \big(x + 3y + 2z\,,\, 2z + 5y + 3z\big)$ and let $(a, b) = (1, 1)$. Find a parametric equation for the level set $f(x, y, z) = (a, b)$.

1.26 Exercise: Let $A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 5 & 2 & -4 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and let $f(x) = Ax + b$. Find an implicit equation for the range of f.

Chapter 2. Differentiation

2.1 Definition: For $a \in \mathbf{R}^n$ and $r > 0$, we define the **open ball** of radius r centred at a to be the set
$$B(a,r) = \left\{ x \in \mathbf{R}^n \Big| |x-a| < r \right\}.$$
For a set $U \subseteq \mathbf{R}^n$, we say that U is **open** in $\mathbf{R}^n$ when for every point $a \in U$ there exists a radius $r > 0$ such that $B(a,r) \subseteq U$. Informally, a set U is open in $\mathbf{R}^n$ when it does not include any of its boundary points.

2.2 Definition: Let $U \subseteq \mathbf{R}^n$ be open in $\mathbf{R}^n$, let $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$, and let $a \in U$. We say that f is **differentiable** at a when there exists an affine map $L : \mathbf{R}^m \to \mathbf{R}^n$ such that
$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in U \Big(|x-a| \le \delta \Longrightarrow \big| f(x) - L(x) \big| \le \epsilon |x-a| \Big).$$
We say that f is differentiable in U when f is differentiable at every point $a \in U$.

2.3 Definition: Let $U \subseteq \mathbf{R}^n$ be open in $\mathbf{R}^n$, let $f : U \subseteq \mathbf{R}^n \to \mathbf{R}$, and let at $a \in U$, say $a = (a_1, \cdots, a_n)$. We define the k^{th} **partial derivative** of f at a to be
$$\tfrac{\partial f}{\partial x_k}(a) = {g_k}'(a_k) \text{ , where } g_k(t) = f(a_1, \cdots, a_{k-1}, t, a_{k+1}, \cdots a_n) \,,$$
or equivalently,
$$\tfrac{\partial f}{\partial x_k}(a) = {h_k}'(0) \text{ , where } h_k(t) = f(a_1, \cdots, a_{k-1}, a_k + t, a_{k+1}, \cdots a_n) \,,$$
provided that the derivatives exist. Note that g_k and h_k are functions of a single variable.

Sometimes $\frac{\partial f}{\partial x_k}$ is written as f_{x_k} or as f_k. When we write $u = f(x)$, we can also write $\frac{\partial f}{\partial x_k}$ as $\frac{\partial u}{\partial x_k}$, u_{x_k} or u_k. When $n = 3$ and we write x, y and z instead of x_1, x_2 and x_3, the partial derivatives $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$ and $\frac{\partial f}{\partial x_3}$ are written as $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$, or as f_x, f_y and f_z. When $n = 1$ so there is only one variable $x = x_1$ we have $\frac{\partial f}{\partial x}(a) = \frac{df}{dx}(a) = f'(a)$.

2.4 Note: To calculate the partial derivative $\frac{\partial f}{\partial x_k}(x)$, we can treat the variables x_i with $i \neq k$ as constants, and differentiate f as if it were a function of the single variable x_k.

2.5 Exercise: Let $f(x,y) = x^3 y + 2xy^2$. Find $\frac{\partial f}{\partial x}(1,2)$ and $\frac{\partial f}{\partial y}(1,2)$.

2.6 Exercise: Let $f(x,y,z) = (x - z^2)\sin(x^2 y + z)$. Find $\frac{\partial f}{\partial x}(x,y,z)$ and $\frac{\partial f}{\partial x}(3, \frac{\pi}{2}, 0)$.

2.7 Definition: Let $U \subseteq \mathbf{R}^n$ be open in $\mathbf{R}^n$, let $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ and let $a \in U$. Write $u = f(x) = \big(f_1(x), f_2(x), \cdots, f_m(x)\big)^T$ with $x = (x_1, x_2, \cdots, x_n)^T$. We define the **derivative matrix**, or the **Jacobian matrix**, of f at a to be the matrix
$$Df(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_k}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_k}(a) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_k}(a) \end{pmatrix}$$
and we define the **linearization** of f at a to be the affine map $L : \mathbf{R}^n \to \mathbf{R}^m$ given by
$$L(x) = f(a) + Df(a)(x-a)$$
provided that all the partial derivatives $\frac{\partial f_k}{\partial x_l}(a)$ exist.

2.8 Definition: Let U be open in $\mathbf{R}^n$ and let $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$. We say that f is $\mathcal{C}^1$ in U when all the partial derivatives $\frac{\partial f_k}{\partial f_l}$ exist and are continuous in U. The **second order partial derivatives** of f are the functions

$$\frac{\partial^2 f_j}{\partial x_k \partial x_l} = \frac{\partial\left(\frac{\partial f_j}{\partial x_l}\right)}{\partial x_k}.$$

We also write $\frac{\partial^2 f_j}{\partial {x_k}^2} = \frac{\partial^2 f_j}{\partial x_k \partial x_k}$. We say that f is $\mathcal{C}^2$ when all the partial derivatives $\frac{\partial^2 f_j}{\partial x_k \partial x_l}$ exist and are continuous in U.

2.9 Definition: Let $a \in U$ where U be an open set in $\mathbf{R}$, and let $f : U \subseteq \mathbf{R} \to \mathbf{R}^m$, say $x = f(t) = \big(x_1(t), x_2(t), \cdots, x_m(t)\big)$. Then we write $f'(a) = Df(a)$ and we have

$$f'(a) = Df(a) = \begin{pmatrix} \frac{\partial x_1}{\partial t}(a) \\ \vdots \\ \frac{\partial x_m}{\partial t}(a) \end{pmatrix} = \begin{pmatrix} {x_1}'(a) \\ \vdots \\ {x_m}'(a) \end{pmatrix}.$$

The vector $f'(a)$ is called the **tangent vector** to the curve $x = f(t)$ at the point $f(a)$. In the case that t represents time and $f(t)$ represents the position of a moving point, $f'(a)$ is also called the **velocity** of the moving point at time $t = a$.

2.10 Definition: Let $a \in U$ where U is an open set in $\mathbf{R}^n$ and let $f : U \subseteq \mathbf{R}^n \to \mathbf{R}$. We define the **gradient** of f at a to be the vector

$$\nabla f(a) = Df(a)^T = \left(\frac{\partial f}{\partial x_1}(a), \cdots, \frac{\partial f}{\partial x_n}(a)\right)^T = \begin{pmatrix} \frac{\partial f}{\partial x_1}(a) \\ \vdots \\ \frac{\partial f}{\partial x_n}(a) \end{pmatrix}.$$

2.11 Theorem: *Let $U \subseteq \mathbf{R}^n$ be open, let $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ and let $a \in U$. Then*

(1) If f is differentiable at a then the partial derivatives of f at a all exist, and the affine map L which appears in the definition of the derivative is the linearization of f at a.
(2) If f is differentiable in U then f is continuous in U.
(3) If f is $\mathcal{C}^1$ in U then f is differentiable in U.
(4) If f is $\mathcal{C}^2$ in U then $\frac{\partial^2 f_j}{\partial x_k \partial x_l} = \frac{\partial^2 f_j}{\partial x_l \partial x_k}$ for all j, k, l.

2.12 Note: Let $a \in U$ where U is open in $\mathbf{R}^n$ and let $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ be differentiable at a. The definition of the derivative, together with Part (1) of the above theorem, imply that the function f is approximated by its linearization near $x = a$, that is when $x \cong a$ we have

$$f(x) \cong L(x) = f(a) + Df(a)(x - a)\,.$$

The geometric objects (curves and surfaces etc) Graph(f), Null(f), $f^{-1}(k)$ and Range(f) are all approximated by the affine spaces Graph(L), Null(L), $L^{-1}(k)$ and Range(L). Each of these affine spaces is called the (affine) **tangent space** of its corresponding geometric object: the space Graph(L) is called the (affine) tangent space of the set Graph(f) at the point $\big(a, f(a)\big)$; when $f(a) = 0$, the space Null(L) is called the (affine) tangent space of Null(f) at the point a, and more generally when $f(a) = k$, so that $a \in f^{-1}(k)$, the space $L^{-1}(k)$ is called the (affine) tangent space to $f^{-1}(k)$ at the point a; and the space Range(L) is called the (affine) tangent space of the set Range(f) at the point $f(a)$. When a tangent space is 1-dimensional we call it a **tangent line** and when a tangent space is 2-dimensional we call it a **tangent plane**.

2.13 Exercise: Find an explicit, an implicit and a parametric equation for the tangent line to the curve in $\mathbf{R}^2$ which is defined explicitly by the equation $y = f(x)$, implicitly by the equation $g(x,y) = k$, and parametrically by the equation $(x,y) = \alpha(t) = \big(x(t), y(t)\big)$.

2.14 Exercise: Find an explicit, an implicit, and a parametric equation for the tangent line to the curve in $\mathbf{R}^3$ which is defined explicitly by $(x,y) = f(z) = \big(x(z), y(z)\big)$, implicitly by $u(x,y,z) = k$ and $v(x,y,z) = l$, and parametrically by $(x,y,z) = \alpha(t) = \big(x(t), y(t), z(t)\big)$.

2.15 Exercise: Find an explicit, an implicit and a parametric equation for the tangent plane to the surface in $\mathbf{R}^3$ which is defined explicitly by $z = f(x,y)$, implicitly by $g(x,y,z) = k$, and parametrically by $(x,y,z) = \sigma(s,t) = \big(x(s,t), y(s,t), z(s,t)\big)$.

2.16 Exercise: Find a parametric equation for the tangent line to the helix given by $(x,y,z) = (2\cos t, 2\sin t, t)$ at the point where $t = \frac{\pi}{3}$, and find the point where this tangent line crosses the xz-plane.

2.17 Exercise: Find an explicit equation for the tangent plane to the surface $z = \dfrac{e^{x^2+2xy}}{\sqrt{2+y}}$ at the point $(2,-1)$.

2.18 Exercise: Find an implicit equation for the tangent line to the curve given by $2\sqrt{y+x^2} + \ln(y-x^2) = 6$ at the point $(2,5)$.

2.19 Exercise: Find a parametric equation for the tangent line to the curve of intersection of the paraboloid $z = 2 - x^2 - y^2$ with the cone $y = \sqrt{x^2+z^2}$ at the point $p = (1,1,0)$.

2.20 Exercise: Find an explicit equation for the tangent plane to the surface given by $(x,y,z) = \big(r\cos t\,,\, r\sin t\,,\, \frac{3}{1+r^2}\big)$ at the point where $(r,t) = \big(\sqrt{2}, \frac{\pi}{4}\big)$.

2.21 Theorem: *(The Chain Rule) Let $f : U \subseteq \mathbf{R}^n \to V \subseteq \mathbf{R}^m$, let $g : V \subseteq \mathbf{R}^m \to \mathbf{R}^l$, and let $h(x) = g(f(x))$. If f is differentiable at a and g is differentiable at $f(a)$ then h is differentiable at a and $Dh(a) = Dg(f(a))Df(a)$.*

2.22 Exercise: Let $z = f(x,y) = 4x^2 - 8xy + 5y^2$, $(u,v) = g(z) = \big(\sqrt{z-1}\,,\, 5\ln z\big)$ and $h(x,y) = g\big(f(x,y)\big)$. Find $Dh(2,1)$.

2.23 Exercise: Let $(x,y) = f(r,\theta) = \big(r\cos\theta\,,\, r\sin\theta\big)$, let $z = g(x,y)$ and let $z = h(r,\theta) = g\big(f(r,\theta)\big)$. If $h(r,\theta) = r^2 e^{\sqrt{3}(\theta - \frac{\pi}{6})}$ then find $\nabla g(\sqrt{3}, 1)$.

2.24 Exercise: Let $(x,y,z) = f(s,t)$ and $(u,v) = g(x,y,z)$. Find a formula for $\frac{\partial u}{\partial t}$.

2.25 Definition: Let $a \in U$ where U is an open set in $\mathbf{R}^n$, let $f : U \subseteq \mathbf{R}^n \to R$ be differentiable at a, and let $v \in \mathbf{R}^n$. We define the **directional derivative of** f **at** a **with respect to** v, written as $D_v f(a)$, as follows: pick any differentiable curve $\alpha(t)$ with $\alpha(0) = a$ and $\alpha'(0) = v$ $\big($for example, we could pick $\alpha(t) = a + v\,t\big)$, and define $D_v f(a)$ to be the rate of change of the function f at $t = 0$ as we move along the curve α. To be precise, let $\beta(t) = f\big(\alpha(t)\big)$, note that $\beta'(t) = Df\big(\alpha(t)\big)\alpha'(t)$, and then define $D_v f(a)$ to be

$$\begin{aligned} D_v f(a) &= \beta'(0) \\ &= Df(\alpha(0))\,\alpha'(0) \\ &= Df(a)\,v \\ &= \nabla f(a) \cdot v\,. \end{aligned}$$

Notice that the formula for $D_v f(a)$ does not depend on the choice of the curve $\alpha(t)$. The (directional) **derivative of** f **in the direction of** v is defined to be the $D_w f(a)$ where w is the unit vector in the direction of v which is given by $w = \frac{v}{|v|}$.

2.26 Exercise: Let $f(x, y, z) = x \sin(y^2 - 2xz)$ and let $\alpha(t) = \big(\sqrt{t}\,,\, \frac{1}{2}t\,,\, e^{(t-4)/4}\big)$. Find the rate of change of f as we move along the curve $\alpha(t)$ when $t = 4$.

2.27 Theorem: *Let $f : U \subseteq \mathbf{R}^n \to \mathbf{R}$ be differentiable at $a \in U$. Say $f(a) = b$. The gradient $\nabla f(a)$ is perpendicular to the level set $f(x) = b$, it is in the direction in which f increases most rapidly, and its length is the rate of increase of f in that direction.*

Proof: Let $\alpha(t)$ be a curve in the level set $f(x) = b$, with $\alpha(0) = a$. We wish to show that $\nabla f(a) \perp \alpha'(0)$. Since $\alpha(t)$ lies in the level set $f(x) = b$, we have $f(\alpha(t)) = b$ for all t. Take the derivative of both sides to get $Df(\alpha(t))\alpha'(t) = 0$. Put in $t = 0$ to get $Df(a)\,\alpha'(0) = 0$, that is $\nabla f(a) \cdot \alpha'(0) = 0$. Thus $\nabla f(a)$ is perpendicular to the level set $f(x) = b$.

Next, let u be a unit vector. Then $D_u f(a) = \nabla f(a) \cdot u = |\nabla f(a)| \cos\theta$, where θ is the angle between u and $\nabla f(a)$. So the maximum possible value of $D_u f(a)$ is $|\nabla f(a)|$, and this occurs when $\cos\theta = 1$, that is when $\theta = 0$, which happens when u is in the direction of $\nabla f(a)$.

2.28 Note: Let $a \in U$ where U is an open set in $\mathbf{R}^n$, and let $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ be differentiable. The k^{th} column vector of the derivative matrix $Df(a)$ is the vector

$$f_{x_k}(a) = \tfrac{\partial f}{\partial x_k}(a) = \Big(\tfrac{\partial f_1}{\partial x_k}(a), \cdots, \tfrac{\partial f_m}{\partial x_k}(a)\Big)^T \in \mathbf{R}^m,$$

which is the tangent vector to the curve $\beta_k(t) = f\big(\alpha_k(t)\big)$ at $t = 0$, where α_k is the curve through a in the direction of the standard basis vector e_k given by $\alpha_k(t) = a + te_k$.

The l^{th} column vector of the derivative matrix $Df(a)$ is the vector

$$\nabla f_l(a) = \Big(\tfrac{\partial f_l}{\partial x_1}(a), \cdots, \tfrac{\partial f_l}{\partial x_n}(a)\Big)^T$$

which is orthogonal to the level set $f_l(x) = f_l(a)$, pointing in the direction in which f_l increases most rapidly, and its length is the rate of increase of f_l in that direction.

Chapter 3. Integration of Scalar-Valued Functions

3.1 Remark: For many sets $D \subseteq \mathbf{R}^n$ and many functions $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$ one can define the **integral of** f **on** D as a limit of Riemann sums, but we shall not give the precise definition here. Instead, we quote a theorem which enables us to calculate these integrals.

3.2 Theorem: *When $D = \{x \in \mathbf{R} | a \le x \le b\}$ and $f : D \subseteq \mathbf{R} \to \mathbf{R}$ is continuous, the integral of f on D is written as*

$$\int_D f\ dL = \int_D f(x)\ dL = \int_{x=a}^b f(x)\, dx.$$

When $D = \{(x,y) \in \mathbf{R}^2 \big| a \le x \le b\,,\ g(x) \le y \le h(x)\}$ and $f : D \subseteq \mathbf{R}^2 \to \mathbf{R}$ is continuous, the integral of f on D is given by

$$\int_D f\ dA = \iint_D f(x,y)\ dA = \iint_D f(x,y)\, dx\, dy = \int_{x=a}^b \left(\int_{y=g(x)}^{h(x)} f(x,y)\, dy \right) dx.$$

When $D = \{(x,y) \in \mathbf{R}^2 \big| c \le y \le d\,,\ k(y) \le x \le l(x)\}$ and $f : D \subseteq \mathbf{R}^2 \to \mathbf{R}$ is continuous, the integral of f on D is given by

$$\int_D f\ dA = \iint_D f(x,y)\ dA = \iint_D f(x,y)\, dx\, dy = \int_{y=c}^d \left(\int_{x=k(y)}^{l(y)} f(x,y)\, dx \right) dy.$$

More generally, when $D \subseteq \mathbf{R}^2$ is a union $D = \bigcup\limits_{i=1}^n D_i$ of sets $D_i \subseteq \mathbf{R}^2$ which only overlap along their boundaries, with each set D_i of one of the above two forms, the integral of f on D is

$$\int_D f\ dA = \sum_{i=1}^n \int_{D_i} f\ dA.$$

When $D = \{(x,y,z) \in \mathbf{R}^3 \big| a \le x \le b\,,\ g(x) \le y \le h(x)\,,\ k(x,y) \le z \le l(x,y)\}$ and $f : D \subseteq \mathbf{R}^3 \to \mathbf{R}$ is continuous, the integral of f on D is given by

$$\begin{aligned}\int_D f\ dV &= \iiint_D f(x,y,z)\ dV = \iiint_D f(x,y,z)\ dx\, dy\, dz \\ &= \int_{x=a}^b \left(\int_{y=g(x)}^{h(x)} \left(\int_{z=k(x,y)}^{l(x,y)} f(x,y,z)\, dz \right) dy \right) dx.\end{aligned}$$

There are similar formulas in the case that the roles of x, y and z are permuted. More generally, when $D \subseteq \mathbf{R}^3$ is a union $D = \bigcup\limits_{i=1}^n D_i$ of sets $D_i \subseteq \mathbf{R}^3$ which only overlap along their boundaries, with each set D_i of the above form or of a similar form with x, y and z permuted, the integral of f on D is

$$\int_D f\ dA = \sum_{i=1}^n \int_{D_i} f\ dA.$$

3.3 Note: When $D \subseteq \mathbf{R}^2$, the integral of the constant function 1 on D measures the **area** of the region D and, when $f : D \subseteq \mathbf{R}^2 \to \mathbf{R}$, the integral of f on D measures the **signed volume** of the region between the graph of f and the region D and, in the case that D represents the shape of a flat object and the function $f : D \subseteq D \to \mathbf{R}$ represents its **density** (or the **charge density**), the integral of f on D measures the total **mass** (or **charge**) of the object.

When $D \subseteq \mathbf{R}^3$, the integral of the constant function 1 on D measures the **volume** of the region D and, when D represents the shape of a solid object and $f : D \subseteq \mathbf{R}^3 \to \mathbf{R}$ represents its **density** (or **charge density**), the integral of f on D measures the total **mass** (or **charge**) of the object.

3.4 Exercise: Let D be the triangle in $\mathbf{R}^2$ with vertices at $(0,-1)$, $(2,1)$ and $(2,3)$. Find $\displaystyle\int_D 2xy\, dA$.

3.5 Exercise: Find the volume of the region in $\mathbf{R}^3$ which lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 2x$.

3.6 Exercise: Find the mass of the tetrahedron with vertices at $(0,0,0)$, $(2,0,0)$, $(2,2,0)$ and $(2,2,2)$ given that the density is given by $\rho(x,y,z) = 2xy(3-z)$.

3.7 Definition: For a set $D \subseteq \mathbf{R}^n$, we say that D is **bounded** when there exists a radius $r > 0$ such that $D \subseteq B(0,r)$, and we say that D is **closed** in $\mathbf{R}^n$ when its complement $\mathbf{R}^n \setminus D$ is open in $\mathbf{R}^n$. For an open set $U \subseteq \mathbf{R}^n$, the **closure** of U, denoted by $\overline{U}$, is the smallest closed set which contains U, and the **boundary** of U is the set $\partial U = \overline{U} \setminus U$.

3.8 Definition: Let U and V be open sets in $\mathbf{R}^n$, let $C = \overline{U}$ and $D = \overline{V}$. An **orientation preserving change of coordinates map** from C to D is a continuous map $g : C \to D$ such that the map $g : U \to V$ is invertible and $\mathcal{C}^1$ with $\det\big(Dg(a)\big) > 0$ for all $a \in U$, and a **orientation reversing change of coordinates map** from C to D is a continuous map $g : C \to D$ such that the map $g : U \to V$ is invertible and $\mathcal{C}^1$ with $\det\big(Dg(a)\big) < 0$ for all $a \in U$.

3.9 Example: Three important orientation preserving change of coordinates maps are the **polar coordinates map** in $\mathbf{R}^2$, which is given by

$$(x,y) = g(r,\theta) = \big(r\cos\theta\,,\ r\sin\theta\big) \text{ with } \det Dg(r,\theta) = r\,,$$

the **cylindrical coordinates map** in $\mathbf{R}^3$, which is given by

$$(x,y,z) = g(r,\theta,z) = \big(r\cos\theta\,,\ r\sin\theta\,,\ z\big) \text{ with } \det Dg(r,\theta,z) = r,$$

and the **spherical coordinates map** in $\mathbf{R}^3$, which is given by

$$(x,y,z) = g(r,\phi,\theta) = \big(r\,\sin\phi\,\cos\theta\,,\ r\,\sin\phi\,\sin\theta\,,\ r\,\cos\phi\big) \text{ with } \det Dg(r,\phi,\theta) = r^2\sin\phi.$$

3.10 Theorem: *(Change of Variables) When $D = [a, b] \subseteq \mathbf{R}$, and $g : C \subseteq \mathbf{R} \to D \subseteq \mathbf{R}$ is a change of variables map from C to D given by $x = g(u)$ with inverse $u = h(x)$, and $f : D \subseteq \mathbf{R} \to \mathbf{R}$ is continuous, we have*

$$\int_{x=a}^{b} f(x)\,dx = \int_D f(x)dx = \int_C f\big(g(u)\big)\Big|\det Dg(u)\Big|\;du = \int_{u=h(a)}^{h(b)} f\big(g(u)\big)g'(u)\;du.$$

When $f : D \subseteq \mathbf{R}^2 \to \mathbf{R}$ is continuous and $g : C \subseteq \mathbf{R}^2 \to D \subseteq \mathbf{R}^2$ is a change of variables map from C to D given by $(x, y) = g(u, v)$, we have

$$\iint_D f(x,y)\,dx\,dx = \iint_C f\big(g(u,v)\big)\Big|\det Dg(u,v)\Big|\;du\,dv.$$

When $f : D \subseteq \mathbf{R}^3 \to \mathbf{R}$ is continuous and $g : C \subseteq \mathbf{R}^3 \to D \subseteq \mathbf{R}^3$ is a change of variables map from C to D given by $(x, y, z) = g(u, v, w)$, we have

$$\iiint_D f(x,y,z)\;dx\,dy\,dz = \iiint_C f\big(g(u,v,w)\big)\Big|\det Dg(u,v,w)\Big|\;du\,dv\,dw.$$

3.11 Exercise: Find the area inside the cardioid $r = 2 + 2\cos\theta$.

3.12 Exercise: Find the volume of the region which lies under the graph of $z = e^{-(x^2+y^2)}$.

3.13 Exercise: Find the volume of the region which lies inside the sphere $x^2+y^2+z^2 = 4$ and inside the cylinder $x^2 - 2x + y^2 = 0$.

3.14 Exercise: Find the mass of the ball $x^2 + y^2 + z^2 \leq 4$ given that the density is given by $\rho(x, y, z) = 1 - \frac{1}{2}\sqrt{x^2 + y^2 + z^2}$.

3.15 Definition: Let $n = 2$ or 3, let $\alpha : [a, b] \subseteq \mathbf{R} \to \mathbf{R}^n$ be continuous on $[a, b]$ and $\mathcal{C}^1$ in (a, b), let C be the curve in $\mathbf{R}^n$ which is given parametrically by $(x, y) = \alpha(t)$ or by $(x, y, z) = \alpha(t)$ for $a \leq t \leq b$, and let $f : C \subseteq \mathbf{R}^n \to \mathbf{R}$ be continuous on $C = \text{Range}\,(\alpha)$. Then we write $dL = \big|\alpha'(t)\big|\,dt$ and we define the (curve) **integral of f on C** to be

$$\int_\alpha f\;dL = \int_C f\;dL = \int_{t=a}^{b} f\big(\alpha(t)\big)\big|\alpha'(t)\big|\,dt.$$

When C is a union $C = \bigcup\limits_{k=1}^{m} C_k$ of curves C_k as above, we define $\displaystyle\int_C f dA = \sum_{k=1}^{m} \int_{C_k} f\;dA$.

Let D be the closure of a bounded open set U in $\mathbf{R}^2$, let $\sigma : D \subseteq \mathbf{R}^2 \to \mathbf{R}^3$ be continuous in D and $\mathcal{C}^1$ in U, let S be the surface in $\mathbf{R}^3$ which is given parametrically by $(x, y, z) = \sigma(s, t)$, and let $f : S \subseteq \mathbf{R}^3 \to \mathbf{R}$ be continuous on S. Then we write $dA = \big|\sigma_s \times \sigma_t\big|\,ds\,dt$ and we define the (surface) **integral of f on S** to be

$$\iint_\sigma f\;dA = \iint_S f\;dA = \iint_D f\big(\sigma(s,t)\big)\big|\sigma_s \times \sigma_t\big|\,ds\,dt.$$

where $\sigma_s = \left(\frac{\partial x}{\partial s}(s,t), \frac{\partial y}{\partial s}(s,t), \frac{\partial z}{\partial s}(s,t)\right)^T$ and $\sigma_t = \left(\frac{\partial x}{\partial s}(s,t), \frac{\partial y}{\partial s}(s,t), \frac{\partial z}{\partial s}(s,t)\right)^T$.

When S is a union $S = \bigcup\limits_{k=1}^{m} S_k$ of surfaces S_k as above, we define $\displaystyle\int_S f dA = \sum_{k=1}^{m} \int_{S_k} f\;dA$.

3.16 Note: When C is a curve in $\mathbf{R}^n$ with $n = 2$ or 3, which is given by $(x, y) = \alpha(t)$ or by $(x, y, z) = \alpha(t)$ for $a \leq t \leq b$, the integral of the constant function 1 on C measures the **length** (or **arclength**) of the curve C, and in the case that C represents the shape of a physical object and the function $f : C \subseteq \mathbf{R}^2 \to \mathbf{R}$ represents its density (or charge density), the integral of f on C measures the total **mass** (or **charge**) of the object.

When S is a surface in $\mathbf{R}^3$ which is given by $(x, y, z) = \sigma(s, t)$ for $(s, t) \in D \subseteq \mathbf{R}^2$, the integral of the constant function 1 on S measures the **area** (or **surface-area**) of the surface S, and in the case that S represents the shape of a physical object and the function $f : S \to \mathbf{R}$ represents its **density** (or **charge density**), the integral of f on S measures the total **mass** (or **charge**) of the surface.

3.17 Exercise: Find the arclength of the helix $\alpha(t) = \big(t, \cos t, \sin t\big)$ for $0 \leq t \leq 2\pi$.

3.18 Exercise: Find the surface area of the torus given by

$$(x, y, z) = \sigma(\theta, \phi) = \Big((2 + \cos\phi)\cos\theta\,, (2 + \cos\phi)\sin\theta\,,\ \sin\phi\Big)$$

for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$.

3.19 Exercise: Find the mass of the hollow sphere $x^2 + y^2 + z^2 = 1$ when the density (mass per unit area) is given by $\rho(x, y, z) = 3 - z$.

3.20 Exercise: Find the mass of the curve of intersection of the paraboloid $z = 2 - x^2 - 2y^2$ with the parabolic sheet $z = x^2$, when the density (mass per unit length) is given by $\rho(x, y, z) = |xy|$.

Chapter 4. Integration of Vector-Valued Functions

4.1 Definition: We define vector-valued integrals of vector-valued functions in the most obvious way. For example, when C is a curve in $\mathbf{R}^n$ with $n = 2$ or 3 and $F : C \subseteq \mathbf{R}^n \to \mathbf{R}^m$ is given by $F = (F_1, F_2, \cdots, F_m)$, we define the (vector-valued) **integral of F on C** to be

$$\int_C F\, dL = \left(\int_C F_1\, dL\,,\,\cdots\,,\,\int_C F_m\, dL \right)$$

and when S is a surface in $\mathbf{R}^3$ and $F : S \subseteq \mathbf{R}^3 \to \mathbf{R}^m$ is given by $F = (F_1, F_2, \cdots, F_m)$, we define the (vector-valued) **integral of F on S** to be

$$\iint_S F\, dA = \left(\iint_S F_1\, dA\,,\,\cdots\,,\,\iint_S F_m\, dA \right).$$

4.2 Definition: Let $U \subseteq \mathbf{R}^n$. A **vector field** on U is a function $F : U \subseteq \mathbf{R}^n \to \mathbf{R}^n$. An **integral curve** (or **field line** or **flow line**) for a vector field F on U is a curve in U along which the tangent line at each point is in the direction of the field F.

4.3 Remark: We can draw a picture of a vector field in $U \subseteq \mathbf{R}^2$ by choosing many points $(x, y) \in U$ and, for each point, we draw the vector $F(x, y)$ at the point (x, y). An integral curve will follow the direction of the vectors at all points.

4.4 Exercise: For each of the following vector fields F on $\mathbf{R}^2$, draw a picture of the field $\frac{1}{4}F$ and sketch some integral curves. In Parts (a) and (b), find an equation for the integral curves.

(a) $F(x, y) = (-y, x)$ (b) $F(x, y) = (y, x)$ (c) $F(x, y) = (x + y, x - y)$.

4.5 Definition: Let U be an open set in $\mathbf{R}^3$, let $g : U \subseteq \mathbf{R}^3 \to \mathbf{R}$ be a function and let $F : U \subseteq \mathbf{R}^3 \to \mathbf{R}^3$ be a vector field given by $F = (P, Q, R)$. We write

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}\ ,\ \nabla g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix}\ ,\ \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\ ,\ \nabla \times F = \begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial F}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix},$$

$$\nabla^2 g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} \quad \text{and} \quad \nabla^2 F = \begin{pmatrix} \nabla^2 P \\ \nabla^2 Q \\ \nabla^2 R \end{pmatrix}.$$

∇g is called the **gradient** of g, $\nabla \cdot F$ is called the **divergence** of F, $\nabla \times F$ is called the **curl** of F, $\nabla^2 g$ is called the (scalar) **Laplacian** of g, and $\nabla^2 F$ is called the **vector Laplacian** of F.

4.6 Theorem: *(Vector Identities) Let U be an open set in $\mathbf{R}^3$, let $g : U \subseteq \mathbf{R}^3 \to \mathbf{R}$ be $\mathcal{C}^2$ in U and let $F : U \subseteq \mathbf{R}^3 \to \mathbf{R}^n$ be $\mathcal{C}^2$ in U. Then*

(1) $\nabla \cdot (\nabla g) = \nabla^2 g$,
(2) $\nabla \times (\nabla g) = 0$,
(3) $\nabla \cdot (\nabla \times F) = 0$,
(4) $\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \nabla^2 F$,
(5) $\nabla \cdot (gF) = \nabla g \cdot F + g(\nabla \cdot F)$, and
(6) $\nabla \times (gF) = (\nabla g) \times F + g(\nabla \times F)$.

4.7 Definition: Let U be an open set in $\mathbf{R}^n$ and let $F : U \subseteq \mathbf{R}^n \to \mathbf{R}^n$ be a vector field in U. When there is a function $g : U \subseteq \mathbf{R}^n \to \mathbf{R}$ such that $F = \nabla g$, we say that F is a **conservative** vector field and that g is a **scalar potential** for F. In the case that $n = 3$, when there is a vector field $G : U \subseteq \mathbf{R}^3 \to \mathbf{R}^n$ such that $F = \nabla \times G$ we say that G is a **vector potential** for F.

4.8 Remark: Scalar and vector potentials, if they exist, are not unique. If g is a scalar potential for F then so is the function $g + c$ for any constant $c \in \mathbf{R}$. If G is a vector potential for F then so is the vector field $G + \nabla g$ for any $\mathcal{C}^2$ function g.

4.9 Remark: Let F be a $\mathcal{C}^1$ vector field on an open set U in $\mathbf{R}^3$. If F has a scalar potential, say $F = \nabla g$, then $\nabla \times F = \nabla \times (\nabla g) = 0$. If F has a vector potential, say $F = \nabla \times G$, then $\nabla \cdot F = \nabla \cdot (\nabla \times G) = 0$.

4.10 Theorem: *Let F be a $\mathcal{C}^1$ vector field on an open set U in $\mathbf{R}^3$.*
(1) If $\nabla \times F = 0$ then F has a scalar potential $g : V \subseteq \mathbf{R}^3 \to \mathbf{R}$ in any open ball $V \subseteq U$.
(2) If $\nabla \cdot F = 0$ then F has a vector potential $G : V \subseteq \mathbf{R}^3 \to \mathbf{R}^3$ in any open ball $V \subseteq U$.

4.11 Exercise: Determine which of the vector fields in Exercise 4.3 are conservative.

4.12 Exercise: Let $F(x, y, z) = \big(x^2 + yz\,,\, -2xy - 2yz\,,\, xy + z^2\big)$. Note that $\nabla \cdot F = 0$. Find a vector potential for F.

4.13 Definition: Let $n = 2$ or 3, let $\alpha : [a, b] \subseteq \mathbf{R} \to \mathbf{R}^n$ be continuous on $[a, b]$ and $\mathcal{C}^1$ in (a, b), let C be the curve in $\mathbf{R}^n$ which is given parametrically by $(x, y) = \alpha(t)$ or by $(x, y, z) = \alpha(t)$ for $a \le t \le b$, and let $F : C \subseteq \mathbf{R}^n \to \mathbf{R}^n$ be a continuous vector field on $C = \mathrm{Range}\,(\alpha)$. We write $T = \frac{\alpha'(t)}{|\alpha'(t)|}$, $dL = \big|\alpha(t)\big|\,dt$ and $d\alpha = T\,dL$, and we define the **integral** (or the **circulation**) **of** F **along** C to be

$$\int_\alpha F \cdot d\alpha = \int_\alpha F \cdot T\ dL = \int_C F \cdot T\ dL = \int_{t=a}^{b} F\big(\alpha(t)\big) \cdot \alpha'(t)\ dt.$$

When $\alpha(t) = \big(x(t), y(t)\big)$ and $F(x, y) = \big(P(x, y), Q(x, y)\big)$ we also use the notation

$$\int_\alpha F \cdot d\alpha = \int_\alpha P dx + Q dy,$$

and when $\alpha(t) = \big(x(t), y(t), z(t)\big)$ and $F(x, y, z) = \big(P(x, y, z), Q(x, y, z), R(x, y, z)\big)$ we also write

$$\int_\alpha F \cdot d\alpha = \int_\alpha P dx + Q dy + R dz.$$

When $C = \bigcup\limits_{k=1}^{m} C_k$ where each C_k is a curve as above, $\displaystyle\int_C F \cdot T\ dL = \sum_{k=1}^{m} \int_{C_k} F \cdot T\ dL.$

Let D be the closure of an open set U in $\mathbf{R}^2$, let $\sigma : D \subseteq \mathbf{R}^2 \to \mathbf{R}^3$ be continuous on D and $\mathcal{C}^1$ in U, let S be the surface in $\mathbf{R}^3$ which is given parametrically by $(x,y,z) = \sigma(s,t)$, and let $F : S \subseteq \mathbf{R}^3 \to \mathbf{R}^3$ be continuous on $S = \text{Range}\,(\sigma)$. We write $N = \frac{\sigma_s \times \sigma_t}{|\sigma_s \times \sigma_t|}$, $dA = \left|\sigma_s \times \sigma_t\right|$ and $d\sigma = (\sigma_s \times \sigma_t)\, ds\, dt$, and we define the **integral** (or **flux**) **of** F **across** S to be

$$\int_\sigma F \cdot d\sigma = \iint_\sigma F \cdot N\; dA = \iint_S F \cdot N\; dA = \iint_D F\big(\sigma(s,t)\big) \cdot (\sigma_s \times \sigma_t)\; ds\, dt.$$

When $\sigma(s,t) = \big(x(s,t), y(s,t), z(s,t)\big)$ and $F(x,y,z) = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$ we also write

$$\int_\sigma F \cdot d\sigma = \int_\sigma P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy.$$

When $S = \bigcup\limits_{k=1}^{m} S_k$ where each S_k is a surface as above, $\displaystyle\int_S F \cdot N\; dA = \sum_{k=1}^{m} \int_{S_k} F \cdot N\; dA$.

4.14 Definition: For a curve C in $\mathbf{R}^n$ where $n = 2$ or 3, we say that C is $\mathcal{C}^1$ when it is defined parametrically by a map $\alpha : [a,b] \subseteq \mathbf{R} \to \mathbf{R}^n$ such that α is continuous on $[a,b]$ and α is $\mathcal{C}^1$ with bounded derivative in (a,b). We say that C is **piecewise** $\mathcal{C}^1$ when it is a union $C = \bigcup_{k=1}^m C_k$ of $\mathcal{C}^1$ curves C_k.

For a surface S in $\mathbf{R}^3$, we say that S is $\mathcal{C}^1$ when it is defined parametrically by a map $\sigma : D = \overline{U} \subseteq \mathbf{R}^2 \to \mathbf{R}^3$, where U is an open set in $\mathbf{R}^2$, such that σ is continuous in D and σ is $\mathcal{C}^1$ with bounded partial derivatives in U, and we say that S is **piecewise** $\mathcal{C}^1$ when it is a union $S = \bigcup_{k=1}^m S_k$ of $\mathcal{C}^1$ surfaces S_k.

When S is a $\mathcal{C}^1$ surface in $\mathbf{R}^3$ given parametrically by $\sigma : D = \overline{U} \subseteq \mathbf{R}^2 \to \mathbf{R}^3$, the **boundary curve** of S is the curve $\partial S = \sigma(\partial U)$. In practice, the boundary curve ∂S is often piecewise $\mathcal{C}^1$.

4.15 Remark: It can be shown, using the Change of Variables Theorem, that the integral of a vector field along a $\mathcal{C}^1$ curve, or across a $\mathcal{C}^1$ surface, does not depend, except perhaps for a sign change, on the choice of parametric equation for the curve or surface. For a curve, the sign depends on the direction we travel along the curve, that is on the direction of the tangent vector T, and for a surface, the sign depends on whether the normal vector N lies on one side of the tangent plane or the other. For a piecewise $\mathcal{C}^1$ curve $C = \bigcup_{k=1}^m C_k$ (or a piecewise $\mathcal{C}^1$ surface $S = \bigcup_{k=1}^m S_k$) the integral of a vector field along C (or across S) depends on the direction in which we move along each curve C_k (or the direction of the normal vector to each surface S_k).

4.16 Note: When $\alpha(t)$ represents the position of an object which moves along the curve C and the vector field F represents the **force** at each point on the curve C, the integral of F along C measures the **work** done by the force on the object along the curve.

When S represents the shape of a surface in space, and F represents the **velocity field** of a fluid which moves through the surface S, the flux of F across S measures the **rate** (the volume per unit time) at which the fluid flows across the surface S, with the sign of the flux indicating whether the fluid flows in the direction of the normal vector N or in the opposite direction.

4.17 Exercise: Let $F(x,y) = (-y,x)$, let $\alpha(t) = (\cos t, \sin t)$ for $0 \le t \le \frac{3\pi}{2}$, and let $\beta(t) = (2-t, 1+2t)$ for $0 \le t \le 2$. Find the integrals $\displaystyle\int_\alpha F \cdot T\; dL$ and $\displaystyle\int_\beta F \cdot T\; dL$.

4.18 Exercise: Let $F(x,y) = \left(\frac{-y}{x^2+y^2}\,,\,\frac{x}{x^2+y^2}\right)$ and let $\alpha(t) = \big(r(t)\cos\theta(t)\,,\,r(t)\sin\theta(t)\big)$ for $a \le t \le b$. Find $\displaystyle\int_\alpha F \cdot T\; dL$. In particular, find $\displaystyle\int_C F \cdot T\; dL$ when C is the line segment from $(2,1)$ to $(1,3)$.

4.19 Exercise: Let $F(x,y,z) = (-xy, z, x^2)$. Find the flux of F across the portion of the paraboloid $z = x^2 + y^2$ which lies above the square given by $-1 \le x \le 1$ and $-1 \le y \le 1$.

4.20 Theorem: *(The Conservative Field Theorem) Let U be an open set in $\mathbf{R}^3$, let C be a piecewise $\mathcal{C}^1$ curve from p to q in U, let $f : U \subseteq \mathbf{R}^3 \to \mathbf{R}$ be $\mathcal{C}^1$ in U, and let $F = \nabla f$. Then*

$$\int_C F \cdot T\; dL = f(q) - f(p).$$

4.21 Theorem: *(Green's Theorem) Let C be a piecewise $\mathcal{C}^1$ curve in $\mathbf{R}^2$ which goes once, counterclockwise, around the boundary $C = \partial U$ of a bounded open set U in $\mathbf{R}^2$. Let $F = (P,Q)$ be a continuous vector field on $D = \overline{U}$ which is $\mathcal{C}^1$ with bounded partial derivatives in U. Then*

$$\int_C F \cdot T\; dL = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\; dA.$$

4.22 Theorem: *(The Divergence Theorem, or Gauss' Theorem) Let S be a piecewise $\mathcal{C}^1$ surface in $\mathbf{R}^3$ which envelopes the boundary $S = \partial U$ of a bounded open set U in $\mathbf{R}^3$, wrapping once around U with the normal vector N pointing outwards. Let F be a continuous vector field on $D = \overline{U}$ which is $\mathcal{C}^1$ with bounded partial derivatives in U. Then*

$$\iint_S F \cdot N\; dA = \iiint_D \nabla \cdot F\; dV.$$

4.23 Theorem: *(Stokes' Theorem) Let S be a $\mathcal{C}^1$ surface in $\mathbf{R}^3$ given parametrically by $\sigma : D = \overline{U} \subseteq \mathbf{R}^2 \to \mathbf{R}^3$ where U is open in $\mathbf{R}^2$. Let C be a piecewise $\mathcal{C}^1$ curve in $\mathbf{R}^3$ which wraps once around the boundary curve $C = \partial S$ in the direction compatible with the right hand rule (when the fingers of the right hand point in the direction of the tangent vector T to the curve, the thumb points in the direction of the normal vector N to the surface). Let F be a continuous vector field on S such that $F\big(\sigma(s,t)\big)$ is $\mathcal{C}^1$ with bounded partial derivatives in U. Then*

$$\int_C F \cdot T\; dL = \iint_S (\nabla \times F) \cdot N\; dA.$$

4.24 Exercise: Let C be the circle $x^2+y^2 = 1$, let D be the disc $D = \big\{(x,y)\big|x^2+y^2 \le 1\big\}$, and let $F(x,y) = (x^2y, -xy^2)$. Verify that the conclusion of Green's Theorem holds.

4.25 Exercise: Let D be the tetrahedron with vertices at $(0,0,0,\ (1,0,0)$, $(0,2,0)$ and $(0,0,2)$ and let S be the boundary surface of D. Let $f(x,y,z) = xy + z^2$ and let $F = \nabla f$. Verify the conclusion of Gauss' Theorem.

4.26 Exercise: Let C be the curve given by $z = x^2$ and $x^2 + y^2 = 1$, let S be the surface given by $z = x^2$ with $x^2 + y^2 \le 1$, and let $F(x,y,z) = (y, -x, z^2)$. Verify the conclusion of Stokes' Theorem.

4.27 Exercise: Let $(x,y)=\alpha(t)$ be a $\mathcal{C}^1$ curve which goes once, counterclockwise, around the boundary $C=\partial U$ of an open set U in $\mathbf{R}^2$ and let $D=\overline{U}$. Show that the area of D is given by

$$A=\int_D dA=\int_\alpha x\,dy-y\,dx.$$

4.28 Exercise: Find the circulation of F along C when $F(x,y)=\big(x-y^3\,,\,x^3+y^3\big)$ and C is the boundary curve of the quarter-disc given by $x\geq 0$, $y\geq 0$ and $x^2+y^2\leq 1$.

4.29 Exercise: Find the flux of F across S when $F(x,y,z)=\big(xy^2\,,\,x^2y\,,\,(x^2+y^2)z^2\big)$ and S is the boundary surface of the cylinder given by $(x,y,z)=\big(\sin t,0,\cos t\big)$ for $0\leq t\leq 2\pi$.

4.30 Exercise: Find the circulation of F along C when F is the vector field given by $F(x,y,z)=\big(x^2z+\sqrt{x^3+x^2+2}\,,\,xy\,,\,xy+\sqrt{z^3+z^2+2}\big)$ and C is the circle given by $y=0$ and $x^2+z^2=1$.

4.31 Note: Let U be an open set in $\mathbf{R}^3$ and let F be a $\mathcal{C}^1$ vector field in U. If $\nabla\times F=0$ in U then it follows from Stokes' Theorem that

(1) $\displaystyle\int_C F\cdot T\;dL=0$ for every loop C which is the boundary curve of a surface in U.

(2) $\displaystyle\int_C F\cdot T\;dL=\int_D F\cdot T\;dL$ whenever C and D are curves from p to q in U such that there exists a continuous deformation of curves in U from C to D which fixes the points p and q, and

(3) $\displaystyle\int_C F\cdot T\;dL=\int_D F\cdot T\;dL$ whenever C and D are loops in U such that there exists a continuous deformation of loops in U from C to D.

4.32 Definition: A surface in $\mathbf{R}^3$ is called **closed** when it is the boundary surface of some bounded open set in $\mathbf{R}^3$.

4.33 Note: Let U be an open set in $\mathbf{R}^3$ and let F be a $\mathcal{C}^1$ vector field in U. If $\nabla\cdot F=0$ in U then it follows from the Divergence Theorem that

(1) $\displaystyle\iint_S F\cdot N\;dA=0$ for every surface S which is the boundary of a region in U,

(2) $\displaystyle\iint_S F\cdot N\;dA=\iint_T F\cdot N\;dA$ whenever S and T are surfaces in U with the same boundary curve $C=\partial S=\partial T$ such that there exists a continuous deformation of surfaces in U from S to T in U which fixes C, and

(3) $\displaystyle\iint_S F\cdot N\;dA=\iint_T F\cdot N\;dA$ whenever S and T are closed surfaces in U such that there exists a continuous deformation of closed surfaces in U from S to T.

4.34 Exercise: The electric field surrounding a long thin vertical wire along the z-axis, with charge density (charge per unit length) ρ, is given by

$$E(x,y,z)=2k\rho\Big(\tfrac{x}{x^2+y^2}\,,\,\tfrac{y}{x^2+y^2}\,,0\Big).$$

Find the work done by the electric field on a small object of unit charge when it moves along the line segment from the point $(1,0,1)$ to the point $(0,2,4)$.

4.35 Exercise: Find the flux of F across S when $F(x,y,z)=\big(x+z^2\,,\,0\,,\,-z-3\big)$ and S is the portion of the ellipsoid $x^2+y^2+3z^2=4$ with $z\leq 1$.

4.36 Theorem: *(Divergence as a Flux Density and Curl as a Circulation Density) Let F be a $\mathcal{C}^1$ vector field in an open set U in $\mathbf{R}^3$ and let $a \in U$. Then*

(1) When D is the closed ball of radius r centred at a and S is the boundary sphere of D, we have

$$(\nabla \cdot F)(a) = \lim_{r \to 0} \frac{1}{\mathrm{Vol}(D)} \iint_S F \cdot N \, dA.$$

(2) When D the disc of radius r centered at a with normal vector N and C is the boundary circle of D, we have

$$(\nabla \times F)(a) \cdot N = \lim_{r \to 0} \frac{1}{\mathrm{Area}(D)} \int_C F \cdot T \, dL.$$

Chapter 5. Maxwell's Equations

5.1 Note: The **Lorentz Force Law** states that when a small object of charge q moves at velocity v, the force exerted on the object by an electric field E and a magnetic field B is given by

$$F = q\big(E + v \times B\big).$$

Electric and magnetic fields, in turn, are produced by charges and currents and they influence each other.

5.2 Definition: **Electrostatics** is the study of electric fields which are produced by a static charge distribution.

5.3 Note: Coulomb's Law states that for a small object of charge q at position $s \in \mathbf{R}^3$. the **electric field** $E(r)$ and the **electric potential** $u(r)$ at the point $r \in \mathbf{R}^3$ are given by

$$E(r) = \frac{q}{4\pi\epsilon_0} \cdot \frac{(r-s)}{|r-s|^3} \quad \text{and} \quad u(r) = \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{|r-s|}.$$

For a static charge distribution on a curve C in $\mathbf{R}^3$ of charge density (charge per unit length) ρ we have

$$E(r) = \int_C \frac{\rho}{4\pi\epsilon_0} \cdot \frac{(r-s)}{|r-s|^3}\, dL \quad \text{and} \quad u(r) = \int_C \frac{\rho}{4\pi\epsilon_0} \cdot \frac{1}{|r-s|}\, dL\,.$$

For a static charge distribution on a surface S in $\mathbf{R}^3$ of charge density (charge per unit area) ρ we have

$$E(r) = \iint_S \frac{\rho}{4\pi\epsilon_0} \cdot \frac{(r-s)}{|r-s|^3}\, dA \quad \text{and} \quad u(r) = \iint_S \frac{\rho}{4\pi\epsilon_0} \cdot \frac{1}{|r-s|}\, dA.$$

For a static charge distribution in a region $D \subseteq \mathbf{R}^3$ of charge density (charge per unit volume) ρ, we have

$$E(r) = \iiint_D \frac{\rho}{4\pi\epsilon_0} \cdot \frac{(r-s)}{|r-s|^3}\, dV \quad \text{and} \quad u(r) = \iiint_D \frac{\rho}{4\pi\epsilon_0} \cdot \frac{1}{|r-s|}\, dV\,.$$

5.4 Exercise: Show that the electric field and electric potential surrounding a long straight wire along the z-axis with charge density (charge per unit length) ρ are given by

$$E(x,y,z) = \frac{\rho}{2\pi\epsilon_0}\left(\frac{x}{x^2+y^2}\,,\,\frac{y}{x^2+y^2}\,,\,0\right)\ ,\ u(x,y,z) = -\frac{\rho}{4\pi\epsilon_0}\,\ln(x^2+y^2).$$

5.5 Exercise: Find the electric field and the electric potential at all points inside and around the solid ball $D = \big\{(x,y,z)\big|x^2+y^2+z^2 \le R\big\}$ with constant charge density (charge per unit volume) ρ at all points in D.

5.6 Theorem: *For a static charge distribution on a curve, surface or region, the electric field and the electric potential are related by*

$$E = -\nabla u.$$

It follows that

$$\nabla \times E = 0.$$

5.7 Theorem: *(Gauss' Law) Let D be the closure of a bounded open set in $\mathbf{R}^3$ and let $S = \partial D$ be its boundary surface. Then for a static charge distribution in $\mathbf{R}^3$, we have*

$$\iint_S E \cdot N \, dA = \iiint_D \frac{\rho}{\epsilon_0} \, dV = \frac{Q}{\epsilon_0}$$

where Q is the total charge in D.

5.8 Corollary: *For a static charge distribution in $\mathbf{R}^3$ we have*

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}.$$

5.9 Exercise: Redo exercises 5.3 and 5.4 using Gauss' Law.

5.10 Definition: The differential equations

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla \times E = 0$$

are called **Maxwell's Equations of Electrostatics**.

5.11 Definition: Magnetostatics is the study of magnetic fields which are produced by a steady state current.

5.12 Note: Let C be a curve which lies on a surface S in $\mathbf{R}^3$. When a current distribution flows along the surface S with **current density** (vector-valued current per unit cross-sectional length) J, the **current** (charge per unit time) which flows across the curve C is given by

$$I = \int_C J \cdot M \, dL$$

where M is a unit vector which is tangent to S and normal to C (we can take $M = N \times T$ where N is the unit normal vector for S and T is the unit normal vector for C).

Let S be a surface in $\mathbf{R}^3$. When a current distribution flows in $\mathbf{R}^3$ with **current density** (vector-valued current per unit cross-sectional area) J, the **current** (charge per unit time) which flows across the surface S is given by

$$I = \iint_S J \cdot N \, dA.$$

5.13 Exercise: A hollow plexiglass sphere of radius R centred at the origin carries a uniform charge distribution of charge density (charge per unit area) ρ, and it rotates about the z-axis at a rate of ω radians per unit time. Find the resulting current density J at each point on the sphere, and calculate the current (charge per unit time) which crosses the curve given by $(x,y,z) = \alpha(t) = \big(R\sin t, 0, R\cos t\big)$ for $0 \leq t \leq \pi$.

5.14 Note: The **Biot-Savard Law** states that the element of magnetic field $dB(r)$ and the element of magnetic potential $dA(r)$ at $r \in \mathbf{R}^3$, which are produced by a small element of current $dI = I\,T\,dL = I\,\alpha'(t)\,dt$ flowing along the curve given by $s = \alpha(t)$, are given by

$$dB(r) = \frac{\mu_0}{4\pi} \cdot \frac{dI \times (r-s)}{|r-s|^3} \quad \text{and} \quad dA(r) = \frac{\mu_0}{4\pi} \cdot \frac{dI}{|r-s|}\,.$$

For a steady current I flowing along the curve C we have

$$B(r) = \int_C \frac{\mu_0}{4\pi} \cdot \frac{dI \times (r-s)}{|r-s|^3} \quad \text{and} \quad A(r) = \int_C \frac{\mu_0}{4\pi} \cdot \frac{dI}{|r-s|}\,.$$

For a steady current distribution flowing on a surface S with current density (vector-valued current per unit cross-sectional length) J we have

$$B(r) = \iint_S \frac{\mu_0}{4\pi} \cdot \frac{J \times (r-s)}{|r-s|^3}\,dA \quad \text{and} \quad A(r) = \iint_S \frac{\mu_0}{4\pi} \cdot \frac{J}{|r-s|}\,dA\,.$$

For a steady current distribution flowing in a region $D \subseteq \mathbf{R}^3$ with current density (vector-valued current per unit cross-sectional area) J we have

$$B(r) = \iiint_D \frac{\mu_0}{4\pi} \cdot \frac{J \times (r-s)}{|r-s|^3}\,dV \quad \text{and} \quad A(r) = \iiint_D \frac{\mu_0}{4\pi} \cdot \frac{J}{|r-s|}\,dV\,.$$

5.15 Exercise: Show that the magnetic field and magnetic potential surrounding a long straight wire along the z-axis carrying the current I are given by

$$B(x,y,z) = \frac{\mu_0 I}{2\pi}\left(\frac{-y}{x^2+y^2}\,,\,\frac{x}{x^2+y^2}\,,\,0\right) \quad \text{and} \quad A(x,y,z) = -\frac{\mu_0 I}{4\pi}\Big(0,0,\ln(x^2+y^2)\Big).$$

5.16 Exercise: Find the magnetic field and magnetic potential at each point along the z-axis, produced by a square loop of wire in the xy-plane which follows the boundary of the region $D = \big\{(x,y)\big| -a \le x \le a, -a \le y \le a\big\}$ and carries a constant current I.

5.17 Exercise: Find the magnetic field and the magnetic potential at all points (x,y,z) inside a long cylindrical wire of radius R centred along the z-axis whose current density is given by $J(u,v,w) = a e^{-k(u^2+v^2)}(0,0,1)$.

5.18 Exercise: Find the magnetic field and the magnetic potential at the origin which is produced by the plexiglass sphere from Exercise 5.13.

5.19 Theorem: *For a steady state current on a curve, surface or region, the magnetic field and the magnetic potential are related by*

$$B = \nabla \times A.$$

It follows that

$$\nabla \cdot B = 0\,.$$

5.20 Theorem: *(Ampère's Circuital Law) Let S be a bounded surface in $\mathbf{R}^3$ and let $C = \partial S$ be its boundary curve. Then for a steady current distribution in $\mathbf{R}^3$ we have*

$$\int_C B \cdot T\;dL = \iint_S \mu_0 J \cdot N\;dA = \mu_0 I$$

where I is the total current flowing through the surface S.

5.21 Corollary: *For a steady current distribution in* $\mathbf{R}^3$ *we have*

$$\nabla \times B = \mu_0 J.$$

5.22 Exercise: Redo Exercises 5.15 and 5.16 using Ampère's Circuital Law.

5.23 Definition: The differential equations

$$\nabla \cdot B = 0 \ \text{ and } \ \nabla \times B = \mu_0 J$$

are called **Maxwell's Equations of Magnetostatics**.

5.24 Note: It has been found, experimentally, that the two equations $\nabla \cdot E = \frac{\rho}{\epsilon_0}$ and $\nabla \cdot B = 0$ both hold even when E and B vary with time. By contrast, the other two equations $\nabla \times E = 0$ and $\nabla \times B = \mu_0 J$ need to be modified.

5.25 Note: Faraday's Law states that, when S is a bounded surface in $\mathbf{R}^3$ and $C = \partial S$ is the boundary curve then, for any charge and current distributions in $\mathbf{R}^3$, we have

$$\int_C E \cdot T \, dL = -\frac{\partial}{\partial t} \iint_S B \cdot N \, dA.$$

5.26 Corollary: *For any charge and current distribution in* $\mathbf{R}^3$ *we have*

$$\nabla \times E = -\frac{\partial B}{\partial t}.$$

5.27 Theorem: *(The Continuity Equation) For any charge and current distribution in* $\mathbf{R}^3$ *we have*

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}.$$

5.28 Note: When the magnetostatics equation $\nabla \times B = \mu_0 J$ holds, we have

$$0 = \nabla \cdot (\nabla \times B) = \nabla \cdot (\mu_0 J) = \mu_0 \nabla \cdot J = -\mu_0 \frac{\partial \rho}{\partial t}$$

so that $\frac{\partial \rho}{\partial t} = 0$. Thus the magnetostatics equation $\nabla \times B = \mu_0 J$ cannot possibly hold when the charge density ρ varies with time.

Since $\nabla \cdot J = -\frac{\partial \rho}{\partial t}$ and $\nabla \cdot E = \frac{\rho}{\epsilon_0}$, we have

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\big(\epsilon_0 \nabla \cdot E\big) = -\epsilon_0 \nabla \cdot \frac{\partial E}{\partial t}$$

so that

$$\nabla \cdot \left(J + \epsilon_0 \frac{\partial E}{\partial t}\right) = 0.$$

This observation led Maxwell to propose that we replace the equation $\nabla \times B = \mu_0 J$ by the equation

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}.$$

5.29 Definition: The four differential equations

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0 \, J + \epsilon_0 \mu_0 \, \frac{\partial E}{\partial t}.$$

are called **Maxwell's Equations of Electromagnetism**. It has been found, experimentally, that these equations hold to a high degree of accuracy.