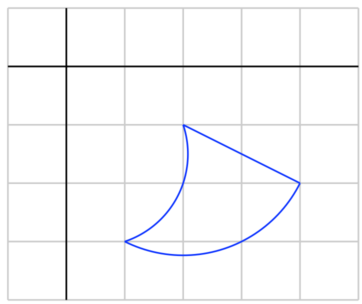
ECE 206 Advanced Calculus 2, Solutions to Assignment 9

1: Sketch the image under $f(z) = 10/z$ of the triangle with vertices $2+i$, $4+2i$ and $1+3i$.

Solution: The line from $2+i$ to $4+2i$ goes through the origin, so it is mapped by $\frac{1}{z}$ to the line through the origin and the point $\frac{1}{2+i} = \frac{2-i}{5}$. On the line from $4+2i$ to $1+3i$, the nearest point to the origin is the point $1+3i$, so this line is mapped by $\frac{1}{z}$ to the circle with diameter 0, $\frac{1}{1+3i} = \frac{1-3i}{10}$. On the line from $1+3i$ to $2+i$, the nearest point to the origin is the point $2+i$, so this line is mapped by $\frac{1}{z}$ to the circle with diameter 0, $\frac{1}{2+i} = \frac{2-i}{5}$. The map $f(z) = \frac{10}{z}$ scales by an additional factor of 10, so it sends the first line to the line through 0 and $4-2i$, it sends the second line to the circle with diameter 0, $1-3i$, and it sends the third line to the circle with diameter 0, $4-2i$.



2: (a) Evaluate $\tanh\left(\ln 2 + i\,\frac{\pi}{4}\right)$.

Solution: We have

$$\tanh\left(\ln 2 + i\,\tfrac{\pi}{4}\right) = \frac{e^{\ln 2 + i\,\pi/4} - e^{-\ln 2 - i\,\pi/4}}{e^{\ln 2 + i\,\pi/4} + e^{-\ln 2 - i\,\pi/4}} = \frac{2e^{i\,\pi/4} - \frac{1}{2}e^{-i\,\pi/4}}{2e^{i\,\pi/4} + \frac{1}{2}e^{-i\,\pi/4}} = \frac{4e^{i\,\pi/2} - 1}{4e^{i\,pi/2} + 1}$$

$$= \frac{-1+4i}{1+4i} = \frac{(-1+4i)(1-4i)}{(1+4i)(1-4i)} = \frac{15+8i}{17}\,.$$

(b) Solve $\tanh z = \tanh i\,z$.

Solution: We have

$$\begin{aligned}
\tanh z = \tanh iz \iff\ & \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \\
\iff\ & \left(e^z - e^{-z}\right)\left(e^{iz} + e^{-iz}\right) = \left(e^z + e^{-z}\right)\left(e^{iz} - e^{-iz}\right) \\
\iff\ & e^{(1+i)z} + e^{(1-i)z} - e^{(-1+i)z} - e^{(-1-i)z} = e^{(1+i)z} - e^{(1-i)z} + e^{(-1+i)z} - e^{-(1-i)z} \\
\iff\ & 2\,e^{(1-i)z} = 2\,e^{(-1+i)z} = 0 \iff e^{2(1-i)z} = 1 \iff 2(1-i)z = 2\pi k\,i \text{ for some } k \in \mathbf{Z} \\
\iff\ & z = \frac{\pi k\,i}{1-i} = \frac{\pi k\,i(1+i)}{2} = \tfrac{\pi}{2}(-1+i)k \text{ for some } k \in \mathbf{Z}\,.
\end{aligned}$$

3: (a) Show that $\cos^{-1} z = -i\,\log\left(z + \sqrt{z^2-1}\right)$, where both sides are multi-fuctions.

Solution: We have

$$\begin{aligned}\cos w = z &\iff \frac{e^{iw}+e^{-iw}}{2} = z \iff e^{iw}+e^{-iw} = 2z \iff (e^{iw})^2 + 1 = 2z(e^{iw}) \\ &\iff (e^{iw})^2 - 2z(e^{iw}) + 1 = 0 \iff e^{iw} = \frac{2z+\sqrt{4z^2-4}}{2} = z + \sqrt{z^2-1} \\ &\iff iw = \log\left(z+\sqrt{z^2-1}\right) \iff w = -i\,\log\left(z+\sqrt{z^2-1}\right).\end{aligned}$$

(b) Solve $\cos z = \frac{1}{4}\,(3+i)$.

Solution: Using the formula from part (a) we have

$$\begin{aligned} z &= \cos^{-1}\left(\tfrac{1}{4}(3+i)\right) = -i\,\log\left(\tfrac{1}{4}(3+i) + \sqrt{\tfrac{1}{16}(8+6i)-1}\right) = -i\,\log\left(\tfrac{1}{4}(3+i)+\tfrac{1}{4}\sqrt{-8+6i}\right) \\ &= -i\,\log\left(\tfrac{1}{4}(3+i) \pm \tfrac{1}{4}(1+3i)\right) = -i\,\log(1+i), -i\,\log\left(\tfrac{1-i}{2}\right) = -i\,\log\left(\sqrt{2}\,e^{i\,\pi/4}\right), -i\,\log\left(\tfrac{1}{\sqrt{2}}\,e^{-i\,\pi/2}\right) \\ &= -i\left(\ln\sqrt{2} + i\left(\tfrac{\pi}{4}+2\pi\,k\right)\right), -i\left(\ln\tfrac{1}{\sqrt{2}} + i\left(-\tfrac{\pi}{4}-2\pi\,k\right)\right) = \pm\left(\left(\tfrac{\pi}{4}+2\pi\,k\right) - i\ln\sqrt{2}\right) \text{ with } k\in\mathbf{Z}.\end{aligned}$$

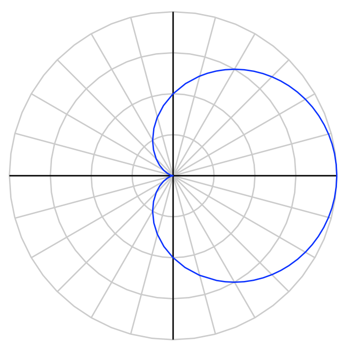
4: (a) Sketch the image under $f(z) = z^2$ of the circle $z(t) = (2\cos t)\,e^{i\,t}$.

Solution: The image is the curve

$$w(t) = f\big(z(t)\big) = \left((2\cos t)e^{i\,t}\right)^2 = (4\cos^2 t)e^{i\,2t} = (2+2\cos 2t)e^{i\,2t}.$$

This is the polar curve $r = 2+2\cos\theta$. We make a table of values and sketch the curve in the polar grid.

$\theta = 2t$	$r = 2+2\cos\theta$
0	4
$\pi/6$	$2+\sqrt{3}$
$\pi/3$	3
$\pi/2$	2
$2\pi/3$	1
$5\pi/6$	$2-\sqrt{3}$
π	0



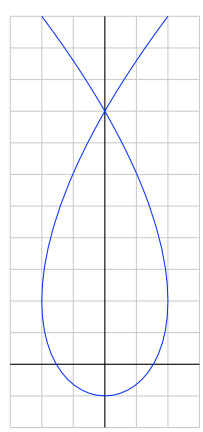
(b) Sketch the image under $f(z) = z^3$ of the line $z(t) = t+i$.

Solution: The line $z(t) = t+i$ is mapped to the curve

$$w(t) = f\big(z(t)\big) = z(t)^3 = (t+i)^3 = t^3 + 3t^2\,i + 3t\,i^2 + i^3 = (t^3-3t) + i\,(3t^2-1) = u(t) + i\,v(t).$$

We make a table of values and sketch the curve.

t	u	v
-2	-2	11
$-\sqrt{3}$	0	8
$-\sqrt{2}$	$\sqrt{2}$	5
-1	2	2
$-1/\sqrt{3}$	$\frac{8\sqrt{3}}{9}$	0
0	0	-1
$1/\sqrt{3}$	$-\frac{8\sqrt{3}}{9}$	0
1	-2	2
$\sqrt{2}$	$-\sqrt{2}$	5
$\sqrt{3}$	0	8
2	2	11



5: Sketch the image under $f(z) = \tanh z$ of the line $z(t) = t + \frac{3\pi}{8}\, i$.

Solution: We provide two solutions. For the first solution, we write

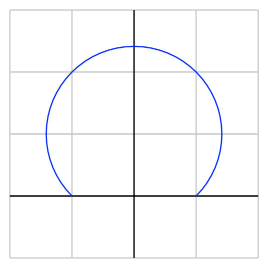
$$\begin{aligned}\tanh(x + i\,y) &= \frac{\sinh(x + i\,y)}{\cosh(x + i\,y)} = \frac{\sinh x \cos y + i\, \cosh x \sin y}{\cosh x \cos y + i\, \sinh x \sin y} \cdot \frac{\cosh x \cos y - i\, \sinh x \sin y}{\cosh x \cos y - i\, \sinh x \sin y} \\ &= \frac{(\sinh x \cosh x \cos^2 y + \sinh x \cosh x \sin^2 y) + i\,(\cosh^2 x \sin y \cos y - \sinh^2 x \sin y \cos y)}{\cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y} \\ &= \frac{(\sinh x \cosh x) + i\,(\sin y \cos y)}{\sinh^2 x + \cos^2 y} = \frac{\left(\frac{1}{2}\sinh 2x\right) + i\,\left(\frac{1}{2}\sin 2y\right)}{\frac{1}{2}\cosh 2x - \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\cos 2y} = \frac{\sinh 2x + i\, \sin 2y}{\cosh 2x + \cos 2y} \\ &= u + i\,v\,,\end{aligned}$$

with $u = \dfrac{\sinh 2x}{\cosh 2x + \cos 2y}$ and $v = \dfrac{\sin 2y}{\cosh 2x + \cos 2y}$. When $x = t$ and $y = \frac{3\pi}{8}$ we have

$$u = \frac{\sinh 2t}{\cosh 2t - \frac{1}{\sqrt{2}}}\ ,\quad v = \frac{\frac{1}{\sqrt{2}}}{\cosh 2t - \frac{1}{\sqrt{2}}}\,.$$

We make a table of values and sketch the curve.

$2t$	u	v
$-\infty$	-1	0
$-\ln\sqrt{2}$	-1	2
$-\ln 2\sqrt{2}$	$-7/5$	$4/5$
0	0	$1+\sqrt{2}$
$\ln\sqrt{2}$	1	2
$\ln 2\sqrt{2}$	$7/5$	$4/5$
∞	1	0



The curve appears to be an arc of the circle $u^2 + (v-1)^2 = 2$, and indeed we have

$$\begin{aligned}u^2 + (v-1)^2 &= \left(\frac{\sinh 2t}{\cosh 2t - \frac{1}{\sqrt{2}}}\right)^2 + \left(\frac{\frac{1}{\sqrt{2}}}{\cosh 2t - \frac{1}{\sqrt{2}}} - 1\right)^2 = \left(\frac{\sinh 2t}{\cosh 2t - \frac{1}{\sqrt{2}}}\right)^2 + \left(\frac{\sqrt{2} - \cosh 2t}{\cosh 2t - \frac{1}{\sqrt{2}}}\right)^2 \\ &= \frac{\sinh^2 2t + 2 - 2\sqrt{2}\cosh 2t + \cosh^2 2t}{\left(\cosh 2t - \frac{1}{\sqrt{2}}\right)^2} = \frac{\cosh^2 2t - 1 + 2 - 2\sqrt{2}\cosh 2t + \cosh^2 2t}{\left(\cosh 2t - \frac{1}{\sqrt{2}}\right)^2} \\ &= \frac{2\left(\cosh^2 2t - \sqrt{2}\cosh 2t + \frac{1}{2}\right)}{\left(\cosh 2t - \frac{1}{\sqrt{2}}\right)^2} = 2\,.\end{aligned}$$

For the second solution, we note that

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1} = 1 - \frac{2}{e^{2z} + 1}$$

and so $w = \tanh z$ is the composite of the maps

$$z_1 = 2z\ ,\quad z_2 = e^{z_1}\ ,\quad z_3 = z_2 + 1\ ,\quad z_4 = \frac{1}{z_3}\ ,\quad z_5 = -2\,z_4\ ,\quad w = z_5 + 1\,.$$

Te map $z_1 = 2z$ sends the line $y = \frac{3\pi}{8}$ to the line $y_1 = \frac{3\pi}{4}$. The map $z_2 = e^{z_1}$ sends the line $y_1 = \frac{3\pi}{4}$ to the ray $z_2 = t\,e^{i\,3\pi/4}$, $t > 0$. The map $z_3 = z_2 + 1$ translates this ray to the ray $1 + t\,e^{i\,3\pi/4}$. The point on this ray nearest the origin is the point $\frac{1+i}{2}$, so the map $z_4 = \frac{1}{z_3}$ sends the ray to the arc, from 1 clockwise to 0, along the circle with diameter 0, $\frac{2}{1+i} = 1 - i$. The map $z_5 = -2z_4$ sends this arc to the arc, from -2 clockwise to 0, along the circle with diameter 0, $-2 + 2i$. Finally, the map $w = z_5 + 1$ sends this to the arc, from -1 clockwise to 1, along the circle with diameter 1, $-1 + 2i$.

6: (a) For $0 \neq a \in \mathbf{C} = \mathbf{R}^2$ and $0 \neq r \in \mathbf{R}$, show that the circle with diameter a, ta has equation

$$|z|^2 - (1+t)\, z \bullet a + t\, |a|^2.$$

Solution: Let $C(a,t)$ be the circle with diameter a, ta. Note that $C(a,t)$ has centre $\frac{a+ta}{2}$ and radius $\left|\frac{a-ta}{2}\right|$ so its equation is

$$\begin{aligned}
\left|z - \tfrac{a+ta}{2}\right|^2 &= \left|\tfrac{a-ta}{2}\right|^2 \\
\left(z - \tfrac{(1+t)a}{2}\right) \bullet \left(z - \tfrac{(1+t)a}{2}\right) &= \tfrac{(1-t)a}{2} \bullet \tfrac{(1-t)a}{2} \\
|z|^2 - (1+t)\, z \bullet a + \tfrac{1}{4}(1+t)^2 |a|^2 &= \tfrac{1}{4}(1-t)^2 |a|^2 \\
|z|^2 - (1+t)\, z \bullet a + t\, |a|^2 &= 0\,.
\end{aligned}$$

(b) For $0 \neq a \in \mathbf{Z}$ and $0 \neq t \in \mathbf{R}$, show that the image under the map $w = f(z) = \frac{1}{z}$ of the circle with diameter a, ta is the circle with diameter $\frac{1}{ta}, \frac{1}{a}$.

Solution: Let $z \in C(a,t)$ so that $|z|^2 - (1+t)\, z \bullet a + t\, |a|^2 = 0$. Then

$$\begin{aligned}
\left|\frac{1}{z}\right|^2 - (1+t)\, \frac{1}{z} \bullet \frac{1}{ta} + t \left|\frac{1}{ta}\right|^2 &= \frac{1}{|z|^2} - (1+t) \frac{\bar{z}}{|z|^2} \bullet \frac{\bar{a}}{|a|^2} + t\, \frac{1}{t^2 |a|^2} \\
&= \frac{1}{|z|^2} - \frac{(1+t)\, z \bullet a}{t\, |z|^2 |a|^2} + \frac{1}{t\, |a|^2} \\
&= \frac{t\, |a|^2 - (1+t)\, z \bullet a + |z|^2}{t\, |z|^2 |a|^2} = 0
\end{aligned}$$

and so $\frac{1}{z} \in C(\frac{1}{ta}, t)$, which is the circle with diameter $\frac{1}{ta}, \frac{1}{a}$. This shows that $f\big(C(a,t)\big) \subseteq C\big(\frac{1}{ta}, t\big)$. Fortunately, we do not need to any additional work to show that $C\big(\frac{1}{ta}, t\big) \subseteq f\big(C(a,t)\big)$ because f is equal to its own inverse. Indeed, if we let $b = \frac{1}{ta}$ so that $C\big(\frac{1}{ta}, t\big) = C(b,t)$, then by our above work, this is sent by $f = f^{-1}$ to the circle $C(\frac{1}{tb}, t) = C(a,t)$, that is $f^{-1}\big(C\big(\frac{1}{ta}, t\big)\big) \subseteq C(a,t)$.