ECE 206 Advanced Calculus 2, Solutions to Assignment 8

**1:** Solve each of the following for $z \in \mathbf{C}$.

(a) $z^3 + 8i = 0$

Solution: Write $z = r\,e^{i\,\theta}$ with $r > 0$ and $\theta \in [0, 2\pi)$. Then

$$\begin{aligned} z^3 + 8i = 0 &\implies \left(r\,e^{i\,\theta}\right)^3 = -8i \implies r^3 e^{i\,3\theta} = 8\,e^{i\,3\pi/2} \\ &\implies r^3 = 8 \text{ and } 3\theta = \tfrac{3\pi}{2} + 2\pi\,k \text{ for some } k \in \mathbf{Z} \\ &\implies r = 2 \text{ and } \theta = \tfrac{\pi}{2} + \tfrac{2\pi}{3}\,k \text{ where } k = 0, 1 \text{ or } 2\,. \end{aligned}$$

Thus the solutions are $z = 2\,e^{i\,\pi/2}$, $z = 2\,e^{i\,7\pi/6}$ and $z = 2\,e^{i\,11\pi/6}$. In cartesian coordinates, the solutions are $z = 2i, -\sqrt{3} - i, \sqrt{3} - i$.

(b) $z^5 + 16\,\overline{z} = 0$

Solution: Let $z = re^{i\,\theta}$. Then we have

$$\begin{aligned} z^5 + 16\overline{z} = 0 &\implies \left(re^{i\,\theta}\right)^5 + 16\,\overline{re^{i\,\theta}} \implies r^5 e^{i\,5\theta} + 16re^{-i\,\theta} = 0 \\ &\implies \left(r = 0 \text{ or } r^4 e^{i\,6\theta} = -16 = 16e^{i\,\pi}\right) \\ &\implies \left(r = 0 \text{ or } \left(r = 2 \text{ and } 6\theta = \pi + 2\pi k\ , \text{ for some } k \in \mathbf{Z}\right)\right) \\ &\implies \left(r = 0 \text{ or } \left(r = 2 \text{ and } \theta = \tfrac{\pi}{6} + \tfrac{\pi}{3}k \text{ for some } k \in \{0,1,2,3,4,5\}\right)\right) \\ &\implies z \in \left\{0, 2e^{i\,\pi/6}, 2e^{i\,\pi/2}, 2e^{i\,5\pi/6}, 2e^{i\,7\pi/6}, 2e^{i\,3\pi/2}, e^{i\,11\pi/6}\right\} \end{aligned}$$

In cartesian coordinates, the solutions are $z = 0, \pm 2i, \pm\sqrt{3} \pm i$.

(c) $2i\,z = \dfrac{z + 2 - i}{z + 1}$

Solution: By the Quadratic Formula, we have

$$\begin{aligned} 2i\,z = \frac{z + 2 - i}{z + 1} &\implies 2i\,z^2 + 2i\,z = z + 2 - i \implies 2i\,z^2 - (1 - 2i)z - (2 - i) = 0 \\ &\implies z = \frac{(1 - 2i) + \sqrt{(1 - 2i)^2 + 4(2i)(2 - i)}}{4i} = \frac{(1 - 2i) + \sqrt{5 + 12i}}{4i} \end{aligned}$$

and $\sqrt{5 + 12i} = \pm\left(\sqrt{\frac{5 + \sqrt{5^2 + 12^2}}{2}} + i\sqrt{\frac{-5 + 5^2 + 12^2}{2}}\right) = \pm\left(\sqrt{\frac{5 + 13}{2}} + i\sqrt{\frac{-5 + 13}{2}}\right) = \pm(3 + 2i)$ so

$$z = \frac{(1 - 2i) \pm (3 + 2i)}{4i} = \frac{4}{4i}, \frac{-2 - 4i}{4i} = -i, -1 + \tfrac{1}{2}\,i\,.$$

**2:** For each of the following polynomials $f(x)$, first solve $f(z) = 0$ for $z \in \mathbf{C}$, and then factor $f(x)$ over the real numbers.

(a) $f(x) = x^6 + 7x^3 - 8$

Solution: For $z \in \mathbf{C}$ we have

$$\begin{aligned} f(z) = 0 &\implies z^6 + 7z^3 - 8 = o \implies (z^3 + 8)(z^3 - 1) = 0 \implies z^3 = 1 \text{ or } z^3 = -8 = 8\,e^{i\,\pi} \\ &\implies z = 1\,,\ e^{i\,2\pi/3}\,,\ e^{i\,4\pi/3}\,,\ 2\,e^{i\pi/3}\,,\ 2\,e^{i\,\pi}\ , \text{ or } 2e^{i\,5\pi/3} \\ &\implies z = 1\,,\ \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\,i\right)\,,\ \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\,i\right)\,,\ \left(1 + \sqrt{3}i\right)\,,\ -2\,,\ \left(1 - \sqrt{3}\,i\right)\,. \end{aligned}$$

It follows that $f(x)$ factors over the reals as $f(x) = (x - 1)(x + 2)(x^2 + x + 1)(x^2 - 2x + 4)$.

(b) $f(x) = x^6 + 1$

Solution: For $z \in \mathbf{C}$ we have

$$\begin{aligned} f(z) = 0 &\implies z^6 + 1 = 0 \implies z^6 = -1 = e^{i\,\pi} \\ &\implies z = e^{i\,\pi/6}\,,\ e^{i\,\pi/2}\,,\ e^{i\,5\pi/6}\,,\ e^{i\,7\pi/6}\,,\ e^{i\,3\pi/2} \text{ or } e^{i\,11\pi/6} \\ &\implies z = \left(\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}\,i\right)\,,\ i\,,\ \left(-\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}\,i\right)\,,\ \left(-\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}\,i\right)\,,\ -i\,,\ \left(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}\,i\right)\,. \end{aligned}$$

It follows that $f(x)$ factors over the reals as $f(x) = (x^2 + 1)(x^2 - \sqrt{3}\,x + 1)(x^2 + \sqrt{3}\,x + 1)$.

**3:** Let $x_1 = 1$ and $x_2 = 3$, and for $n \geq 3$ let $x_n = 4x_{n-1} - 5x_{n-2}$. Find a closed-form formula for $x_n$ (your final answer should not involve complex numbers).

Solution: The associate quadratic equation is $z^2 - 4z + 5 = 0$ with roots $z = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$, so we have $x_n = A(2+i)^n + B(1-2i)^n$ for some constants $A$ and $B$. To get $x_1 = 1$ and $x_2 = 3$ we need $A(2+i) + B(2-i) = 1$ and $A(2+i)^2 + B(2-i)^2 = 3$. Solving these two equations gives $A = \frac{1-3i}{10}$ and $B = \frac{1+3i}{10}$, so we have

$$\begin{aligned} x_n &= \tfrac{1-3i}{10}(2+i)^n + \tfrac{1+3i}{10}(2-i)^n = 2\,\Re\left(\tfrac{1-3i}{10}(2+i)^n\right) \\ &= \tfrac{1}{5}\,\Re\left((1-3i)\left(\sqrt{5}e^{i\,\theta}\right)^n\right) = \tfrac{\sqrt{5}^{\,n}}{5}\Re\left((1-3i)(\cos n\theta + i\sin n\theta)\right) \\ &= \sqrt{5}^{\,n-2}(\cos n\theta + 3\sin n\theta) \end{aligned}$$

where $\theta = \tan^{-1}\frac{1}{2}$.

**4:** Solve each of the following for $x \in \mathbf{R}$. Express your answers using only real numbers.

(a) $8x^3 - 6x + 1 = 0$

Solution: Divide both sides by 8 to get $x^3 - \frac{3}{4}x + \frac{1}{8} = 0$. Let $x = z + \frac{1}{4z}$. Then the equation becomes

$$\begin{gathered} \left(z + \tfrac{1}{4z}\right)^3 - \tfrac{3}{4}\left(z + \tfrac{1}{4z}\right) + \tfrac{1}{8} = 0 \\ z^3 + \tfrac{3}{4}\,z + \tfrac{3}{16z} + \tfrac{1}{64z^3} - \tfrac{3}{4}\,z - \tfrac{3}{16z} + \tfrac{1}{8} = 0 \\ z^3 + \tfrac{1}{8} + \tfrac{1}{64z^3} = 0 \\ z^6 + \tfrac{1}{8}z^3 + \tfrac{1}{64} = 0 \end{gathered}$$

The Quadratic Formula gives

$$z^3 = \frac{-\frac{1}{8} + \sqrt{\frac{1}{64} - \frac{4}{64}}}{2} = -\tfrac{1}{16} \pm \tfrac{\sqrt{3}}{16}\,i = \tfrac{1}{8}\left(-\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2}\,i\right) = \tfrac{1}{8}\,e^{\pm i\,2\pi/3}.$$

When $z^3 = \frac{1}{8}\,e^{i\,2\pi/3}$ we have

$$z = \tfrac{1}{2}\,e^{i\,2\pi/9},\ \tfrac{1}{2}\,e^{i\,8\pi/9} \text{ or } \tfrac{1}{2}\,e^{14\pi/9}.$$

Note that for $z = \frac{1}{2}\,e^{i\,\theta}$ we have $x = z + \frac{1}{4z} = \frac{1}{2}e^{i\,\theta} + \frac{1}{4\cdot\frac{1}{2}e^{-i\,\theta}} = \frac{1}{2}e^{i\,\theta} + \frac{1}{2}e^{-i\,\theta} = \cos\theta$, and so the above three values of $z$ give

$$x = \cos\tfrac{2\pi}{9}, \cos\tfrac{8\pi}{9}, \cos\tfrac{14\pi}{9}.$$

These solutions can also be written as $x = \cos(40°)$, $-\cos(20°)$, $\cos(80°)$.

(b) $x^3 + 3x^2 - 3x - 7 = 0$

Solution: Write $x = y - 1$. Then the equation becomes

$$\begin{gathered} (y-1)^3 + 3(y-1)^2 - 3(y-1) - z = 0 \\ y^3 - 3y^2 + 3y - 1 + 3y^2 - 6y + 3 - 3y + 3 - 7 = 0 \\ y^3 - 6y - 2 = 0 \end{gathered}$$

Write $y = z + 2z^{-1}$. Then the equation becomes

$$\begin{gathered} (z + 2z^{-1})^3 - 6(z + 2z^{-1}) - 2 = 0 \\ z^3 + 6z + 12z^{-1} + 8z^{-3} - 6z - 12z^{-1} - 2 = 0 \\ z^3 + 8z^{-3} - 2 = 0 \\ z^6 - 2z^3 + 8 = 0 \end{gathered}$$

The Quadratic Formula gives $z^3 = \frac{2 \pm \sqrt{4-32}}{2} = 1 \pm \sqrt{7}\,i$. When $z^3 = 1 + \sqrt{7}\,i = 2\sqrt{2}\,e^{i\,\theta}$, with $\theta = \tan^{-1}\sqrt{7}$, we have

$$z = \sqrt{2}\,e^{i\,\theta/3},\ \sqrt{2}\,e^{i\,(\theta+2\pi)/3},\ \text{or } \sqrt{2}\,e^{i\,(\theta+4\pi)/3}.$$

Note that for $z = \sqrt{2}\,e^{i\,\phi}$, we have $y = z + 2z^{-1} = \sqrt{2}e^{i\,\phi} + \sqrt{2}\,e^{-i\,\phi} = 2\sqrt{2}\cos\phi$, so the above three values for $z$ give

$$y = 2\sqrt{2}\cos\left(\tfrac{\theta}{3}\right),\ 2\sqrt{2}\cos\left(\tfrac{\theta+2\pi}{3}\right),\ \text{or } 2\sqrt{2}\cos\left(\tfrac{\theta+4\pi}{3}\right).$$

Finally, since $x = y - 1$ we have

$$x = 2\sqrt{2}\cos\left(\tfrac{\theta}{3}\right) - 1,\ 2\sqrt{2}\cos\left(\tfrac{\theta+2\pi}{3}\right) - 1,\ 2\sqrt{2}\cos\left(\tfrac{\theta+4\pi}{3}\right) - 1,\ \text{where } \theta = \tan^{-1}\sqrt{7}.$$