ECE 206 Advanced Calculus 2, Solutions to Assignment 7

1: Express each of the following complex numbers in cartesian form.

(a) $(2+i)(3+2\,i)-(5+3\,i)$

Solution: We have $(2+i)(3+2i)-(5+3i)=(6+4i+3i-2)-(5+3i)=(4+7i)-(5+3i)=-1+4i$.

(b) $\dfrac{1}{(\sqrt{2}+i)(1+i\sqrt{2})}$

Solution: We have $\dfrac{1}{(\sqrt{2}+i)(1+i\sqrt{2})}=\dfrac{1}{\sqrt{2}+2i+i-\sqrt{2}}=\dfrac{1}{3i}=-\frac{1}{3}\,i$.

(c) $\dfrac{(1+2\,i)^3}{(3+i)^2}$

Solution: We have

$$\begin{aligned}\frac{(1+2\,i)^3}{(3+i)^2}&=\frac{1+6i+12i^2+8i^3}{9+6i+i^2}=\frac{1+6i-12-8i}{9+6i-1}=\frac{-11-2i}{8+6i}\\&=\frac{-11-2i}{2(4+3i)}\cdot\frac{4-3i}{4-3i}=\frac{-44+33i-8i-6}{2(16+9)}=\frac{-50+25i}{50}=-1+\tfrac{1}{2}\,i\,.\end{aligned}$$

2: Solve each of the following equations for $z\in\mathbf{C}$. Express your answers in cartesian form.

(a) $\dfrac{z+1}{z+i}=3+i$

Solution: We have

$$\begin{aligned}&\frac{z+1}{z+i}=3+i\iff(z+1)=(3+i)(z+i)\iff z+1=3z+3i+iz-1\iff 2-3i=(2+i)z\\&\iff z=\frac{2-3i}{2+i}=\frac{2-3i}{2+i}\cdot\frac{2-i}{2-i}=\frac{1-8i}{5}\iff z=\tfrac{1}{5}-\tfrac{8}{5}i\,.\end{aligned}$$

(b) $z^2=\overline{z}$

Solution: Write $z=x+iy$ with $x,y\in\mathbf{R}$. Then we have

$$z^2=\overline{z}\iff(x+iy)^2=x-iy\iff(x^2-y^2)+i(2xy)=x-iy\iff\big(x^2-y^2=x\text{ and }2xy=-y\big)\,.$$

To get $2xy=-y$ we need $y=0$ or $2x=-\frac{1}{2}$. When $y=0$, the equation $x^2-y^2=x$ gives $x^2=x$ so $x=0$ or 1. When $2x=-1$ so $x=-\frac{1}{2}$, the equation $x^2-y^2=x$ gives $\frac{1}{4}-y^2=-\frac{1}{2}$ so $y^2=\frac{3}{4}$, that is $y=\pm\frac{\sqrt{3}}{2}$. Thus $z=x+iy=0,\,1,\,-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ or $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$.

(c) $z^2=4+3\,i$

Solution: This can be solved immediately using the formula from Example 1.13 in the complex analysis lecture notes. Here is an alternate solution. Write $z=x+iy$ with $x,y\in\mathbf{R}$. Then we have

$$z^2=4+3i\iff(x+iy)^2=4+3i\iff(x^2-y^2)+i(2xy)=4+3i\iff\big(x^2-y^2=4\text{ and }2xy=3\big)\,.$$

From $2xy=3$ we get $y=\frac{3}{2x}$. Put this into the equation $x^2-y^2=4$ to get

$$x^2-\frac{9}{4x^2}=4\implies 4x^4-9=16x^2\implies 4x^4-16x^2-9=0\implies(2x^2-9)(2x^2+1)=0\implies\big(x^2=\tfrac{9}{2}\text{ or }x^2=-\tfrac{1}{2}\big)\,.$$

Since $x\in\mathbf{R}$ we must have $x^2=\frac{9}{2}$ so $x=\pm\frac{3}{\sqrt{2}}$. When $x=\frac{3}{\sqrt{2}}$ we have $y=\frac{3}{2x}=\frac{3}{2}\cdot\frac{\sqrt{2}}{3}=\frac{1}{\sqrt{2}}$, and when $x=-\frac{3}{\sqrt{2}}$ we have $y=\frac{3}{2x}=-\frac{1}{\sqrt{2}}$. Thus $z=\pm\left(\frac{3}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,i\right)$.

3: Draw a picture of each of the following subsets of the plane.

(a) $\{z \in \mathbf{C} \big| 1 < |z-1| \leq \sqrt{5}\}$

Solution: We have $1 < |z-1| < \sqrt{5}$ when the distance between z and 1 is greater than 1, so z lies outside the unit circle centred at 1, and is less than or equal to 5, so z lies inside or on the circle of radius $\sqrt{5}$ centred at 1.

(b) $\{z \in \mathbf{C} \big| |z-2\,i| = |z-4|\}$

Solution: We have $|z-2i| = |z-4|$ when z is equidistant from the points 2i and 4, that is when z lies on the perpendicular bisector of the line segment from 2i to 4.

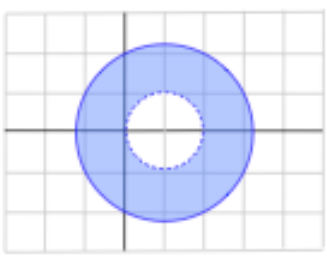
(c) $\{z \in \mathbf{C} \big| |z| + |z-4| = 8\}$

Solution: We have $|z| + |z-4| = 8$ when the sum of the distance from 0 to z with the distance from z to 4 is equal to 8. This happens when z is on the ellipse, with focii at 0 and 4, which passes through the points -2, $\pm 3i$, $4 \pm 3i$, and 6.

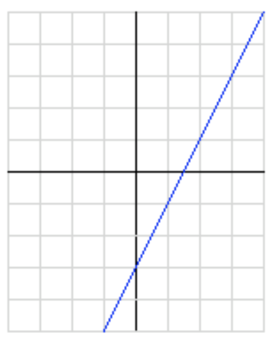
Alternatively, writing $z = x + iy$ with $x, y \in \mathbf{R}$ we can see that this set is the above ellipse as follows.

$$\begin{aligned}
|z| + |z-4| = 8 &\iff \sqrt{x^2+y^2} + \sqrt{(x-4)^2+y^2} = 8 \iff \sqrt{x^2-8x+16+y^2} = 8 - \sqrt{x^2+y^2} \\
&\iff x^2 - 8x + 16 + y^2 = 64 - 16\sqrt{x^2+y^2} + x^2 + y^2 \iff 16\sqrt{x^2+y^2} = 8x + 48 \\
&\iff 2\sqrt{x^2+y^2} = x + 6 \iff 4(x^2+y^2) = x^2 + 12x + 36 \iff 3x^2 - 12x + 4y^2 = 36 \\
&\iff 3(x-2)^2 + 4y^2 = 48 \iff \frac{(x-2)^2}{16} + \frac{y^2}{12} = 1\,.
\end{aligned}$$

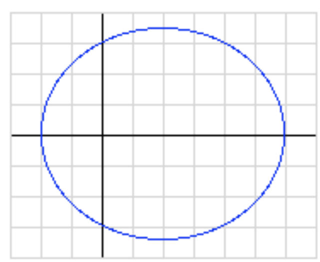
(a)



(b)



(c)



4: Express each of the following complex numbers in cartesian form.

(a) $4\,e^{i\,5\pi/3}$

Solution: We have $4\,e^{i\,5\pi/3} = 4\left(\cos\frac{5\pi}{3} + i\,\sin\frac{5\pi}{3}\right) = 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right) = 2 - 2\sqrt{3}\,i$

(b) $(1+i\sqrt{3})^{10}$

Solution: We have

$$\begin{aligned}(1+i\sqrt{3})^{10} &= \left(2e^{i\,\pi/3}\right)^{10} = 2^{10}e^{i\,10\pi/3} = 2^{10}\left(\cos\tfrac{10\pi}{3} + i\sin\tfrac{10\pi}{3}\right)\\ &= 1024\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\,i\right) = -512 - 512\sqrt{3}\,i\,.\end{aligned}$$

(c) $5\,e^{i\,\theta}$, where $\theta = \tan^{-1} 2$

Solution: We have $5\,e^{i\,\theta} = 5\,(\cos\theta + i\sin\theta) = 5\left(\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}\,i\right) = \sqrt{5} + 2\sqrt{5}\,i.$

5: Express each of the following complex numbers in the polar form $re^{i\,\theta}$.

(a) $-2+2i$

Solution: We have $-2+2i = 2\sqrt{2}\,e^{i\,3\pi/4}$.

(b) $\dfrac{(1-i)^2}{(1+i\sqrt{3})}$

Solution: We have $\dfrac{(1-i)^2}{(1+i\,\sqrt{3})} = \dfrac{\left(\sqrt{2}\,e^{-i\,\pi/4}\right)^2}{2\,e^{i\pi/3}} = \dfrac{2\,e^{-i\,\pi/2}}{2\,e^{i\,\pi/2}} = e^{-i\left(\frac{\pi}{2}+\frac{\pi}{3}\right)} = e^{-i\,5\pi/6}.$

(c) $-3-i$

Solution: We have $-3-i = \sqrt{10}\,e^{i\,\theta}$, where $\theta = \pi + \tan^{-1}\frac{1}{3}$.