ECE 206 Advanced Calculus 2, Solutions to Assignment 6

1: A long cylindrical wire of radius R, centred along the z-axis, carries a uniform charge distribution of charge density (charge per unit volume) ρ. Find the electric field E at all points $(x,y,z)\in\mathbf{R}^3$.

Solution: By symmetry, we assume that E points radially outwards from the z-axis and that the magnitude of E at (x,y,z) depends only on $r=\sqrt{x^2+y^2}$, and we shall write $|E(r)|$ to denote the magnitude of E at all points (x,y,z) with $\sqrt{x^2+y^2}=r$. Let S be the boundary surface of a solid cylinder of length L and radius r centred along the z-axis. Let Φ be the flux of E across S. Since E is always horizontal, we see that the flux across the top and bottom parts of S are equal to zero, so Φ is equal to the flux across the vertical wall of the cylinder. Since E points radially outwards (in the direction of the normal vector to the vertical wall of the cylinder), the flux is

$$\Phi=\iint_S E\bullet N\,dA=\iint_S |E(r)|\,dA=2\pi r\,L|E(r)|.$$

On the other hand, by Gauss' Law we have $\Phi=\frac{Q}{\epsilon_0}$ where Q is the total charge inside S. When $r\geq R$ we have $Q=\rho\,\pi R^2L$ and so

$$|E(r)|=\frac{\Phi}{2\pi rL}=\frac{Q/\epsilon_0}{2\pi rL}=\frac{\rho\,\pi R^2L}{2\pi rL\epsilon_0}=\frac{\rho R^2}{2\epsilon_0 r}$$

and when $r\leq R$ we have $Q=\rho\,\pi r^2L$ and so

$$|E(r)|=\frac{Q/\epsilon_0}{2\pi rL}=\frac{\rho\,\pi r^2L}{2\pi rL\epsilon_0}=\frac{\rho\,r}{2\epsilon_0}.$$

To express E in Cartesian coordinates, we replace r by $\sqrt{x^2+y^2}$ and note that the unit vector which points radially outwards from the z-axis is given by $\frac{1}{\sqrt{x^2+y^2}}(x,y,0)$, and so

$$E(x,y,z)=\begin{cases}\dfrac{\rho R^2}{2\epsilon_0}\Big(\dfrac{x}{x^2+y^2},\dfrac{y}{x^2+y^2},0\Big) & \text{if } x^2+y^2\geq R^2,\text{ and}\\[2ex] \dfrac{\rho}{2\epsilon_0}\big(x,y,0\big) & \text{if } 0<x^2+y^2\leq R.\end{cases}$$

2: (a) A circular loop of wire of radius r lies in the xy-plane centred at the origin. The wire carries a constant current I in the counterclockwise direction (looking down from above). Find the magnetic field B at all points on the z-axis.

Solution: The wire follows the curve C given by $q = (u, v, w) = \alpha(t) = (r\cos t, r\sin t, 0)$ for $0 \le t \le 2\pi$. Writing $p = (0, 0, z)$ and $dI = I\alpha'(t)\,dt$, the Biot-Savard Law gives

$$\begin{aligned} B(p) &= \int_C \frac{\mu_0}{4\pi} \cdot \frac{dI \times (p-q)}{|p-q|^3} \\ &= \int_{t=0}^{2\pi} \frac{\mu_0 I}{4\pi} \cdot \frac{(-r\sin t, r\cos t, 0) \times (-r\cos t, -r\sin t, z)}{\big((r\cos t)^2 + (r\sin t)^2 + z^2\big)^{3/2}}\,dt \\ &= \int_{t=0}^{2\pi} \frac{\mu_0 I}{4\pi} \cdot \frac{(rz\cos t, rz\sin t, r^2)}{(r^2+z^2)^{3/2}}\,dt \\ &= \frac{\mu_0 I}{2} \cdot \frac{r^2}{(r^2+z^2)^{3/2}} \cdot (0,0,1) \end{aligned}$$

since $\displaystyle\int_0^{2\pi} rz\cos t\;dt = 0 = \int_0^{2\pi} rz\sin t\;dt$ and $\displaystyle\int_0^{2\pi} r^2\,dt = 2\pi\,r^2$.

(b) A circular disc of radius R lies in the xy-plane centred at the origin. The disc carries a uniform charge distribution of charge density (charge per unit area) ρ, and it rotates counterclockwise (looking down from above) at a rate of ω radians per unit time. Find the magnetic field B at the origin.

Solution: Imagine that the disc is cut up into thin circles (or annuli) centred at the origin. Each thin circle acts as a circuit carrying charge. Consider a thin circle of radius r and thickness dr. In the time interval dt, the disc turns through the angle $d\theta = \omega\,dt$ radians. An arc of $d\theta$ radians along the thin circle has area $dA = r\,d\theta\,dr = \omega r\;dr\,dt$ and carries a charge $dQ = \rho dA = \rho\omega r\,dr\,dt$, and so the thin circle acts as a circuit of current $dI = \frac{dQ}{dt} = \rho\omega r\,dr$. By Part (a), the thin circle of radius r and thickness dr makes a contribution to the magnetic field at the origin which is vertical with z-component

$$dB_z = \frac{\mu_0 dI}{2} \cdot \frac{r^2}{(r^2+0^2)^{3/2}} = \frac{\mu_0 dI}{2r} = \frac{\mu_0\rho\omega r\,dr}{2r} = \tfrac{1}{2}\mu_0\rho\omega\,dr$$

and the total magnetic field at the origin is vertical with z-component

$$B_z = \int_{z=0}^{R} dB_z = \int_{r=0}^{R} \tfrac{1}{2}\mu_0\rho\omega\,dr = \tfrac{1}{2}\mu_0\rho\omega R.$$

Thus

$$B(0,0,0) = \tfrac{1}{2}\mu_0\rho\omega R\,(0,0,1).$$

3: A long thin straight wire lies along the z-axis and carries a constant current I in the positive z direction. The magnetic field surrounding the wire is given by

$$B(x,y,z) = \frac{\mu_0 I}{2\pi}\Big(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\Big).$$

(a) Find $\displaystyle\int_\sigma B \bullet N \, dA$ where $(x,y,z) = \sigma(s,t) = \big(s, 0, t\big)$ for $1 \le s \le 3$ and $0 \le t \le 2$.

Solution: We have $B\big(\sigma(s,t)\big) = \frac{\mu_0 I}{2\pi}\big(0, \frac{1}{s}, 0\big)$ and $\sigma_s \times \sigma_t = (1,0,0) \times (0,0,1) = (0,-1,0)$ and so

$$\begin{aligned}\int_\sigma B \bullet N \, dA &= \int_{s=1}^{3}\int_{t=0}^{3} \frac{\mu_0 I}{2\pi}\big(0, \tfrac{1}{s}, 0\big) \bullet (0,-1,0)\, dt\, ds \\ &= \int_{s=1}^{3}\int_{t=0}^{2} -\frac{\mu_0 I}{2\pi\, s} dt\, ds = \int_{s=1}^{3} -\frac{\mu_0 I}{\pi s}\, ds = -\frac{\mu_0 I \ln 3}{\pi}.\end{aligned}$$

(b) Find $\displaystyle\int_\alpha B \bullet T \, dL$ where $(x,y,z) = \alpha(t) = \big(4t, t^2-1, t^3\big)$ for $-1 \le t \le 2$.

Solution: As in Problem 6(a) on Assignment 5, if we express $\alpha(t)$ in cylindrical coordinates as $\alpha(t) = \big(r(t)\cos\theta(t), r(t)\sin\theta(t), z(t)\big)$ for $-1 \le t \le 2$ then we have

$$\begin{aligned}B\big(\alpha(t)\big) &= \frac{\mu_0}{2\pi}\Big(-\frac{r\sin\theta}{r^2}, \frac{r\cos\theta}{r^2}, 0\Big) \\ \alpha'(t) &= \big(r'\cos\theta - r\sin\theta\,\theta', r'\sin\theta + r\cos\theta\,\theta', z'\big) \\ B\big(\alpha(t)\big) \bullet \alpha'(t) &= \frac{\mu_0}{2\pi}\Big(-\frac{r'\sin\theta\cos\theta}{r} + \sin^2\theta\,\theta' + \frac{r'\sin\theta\cos\theta}{r} + \cos^2\theta\,\theta'\Big) = \frac{\mu_0 I}{2\pi}\,\theta'\end{aligned}$$

so that

$$\int_\alpha B \bullet T \, dL = \int_{t=-1}^{2} \frac{\mu_0 I}{2\pi}\,\theta'(t)\, dt = \frac{\mu_0 I}{2\pi}\big(\theta(2) - \theta(-1)\big).$$

Since the top view of the curve $\alpha(t)$ is the parabola $(x,y) = \beta(t) = (4t, t^2-1)$ which starts at $\beta(-1) = (-4,0)$ and curves counterclockwise around the origin going through $\beta(0) = (0,-1)$ and $\beta(1) = (4,0)$ and ending at $\beta(2) = (8,3)$, we can take $\theta(-1) = -\pi$, $\theta(0) = -\frac{\pi}{2}$, $\theta(1) = 0$ and $\theta(2) = \tan^{-1}\frac{3}{8}$ and we obtain

$$\int_\alpha B \bullet T \, dL = \frac{\mu_0 I}{2\pi}\big(\theta(2) - \theta(-1)\big) = \frac{\mu_0 I}{2\pi}\big(\tan^{-1}\tfrac{3}{8} + \pi\big).$$

4: The cone given by $z=\sqrt{x^2+y^2}$ with $x^2+y^2\leq 4$ carries a nonuniform charge distribution with charge density (charge per unit area) given by $\rho(x,y,z)=z$. Find the electric field E at the point $(0,0,2)$.

Solution: In general, when a surface S has charge density ρ a small piece of the surface at position $q=(u,v,w)$ of area dA carries a charge of $dQ=\rho dA$ and makes a contribution to the electric field E at the point $p=(x,y,z)$ equal to

$$dE=\frac{\rho\,dA}{4\pi\epsilon_0|p-q|^2}\cdot\frac{p-q}{|p-q|}=\frac{\rho}{4\pi\epsilon_0}\cdot\frac{p-q}{|p-q|^3}\,dA,$$

and the total electric field at p is

$$E=E(p)=\iint_S\frac{\rho}{4\pi\epsilon_0}\cdot\frac{p-q}{|p-q|^3}\,dA.$$

Now let S be the given cone. Note that S is given parametrically by

$$q=(u,v,w)=\sigma(r,\theta)=\big(r\cos\theta,r\sin\theta,r\big)\ \text{ with } 0\leq r\leq 2\,,\ 0\leq\theta\leq 2\pi$$

and we have

$$\big|\sigma_r\times\sigma_\theta\big|=\big|(\cos\theta,\sin\theta,1)\times(-r\sin\theta,r\cos\theta,0)\big|=\big|(-r\cos\theta,-r\sin\theta,r)\big|=\sqrt{2}\,r.$$

By symmetry, the electric field at $p=(0,0,2)$ is vertical. The z-component of E is

$$\begin{aligned}E_z&=\iint_S\frac{w}{4\pi\epsilon_0}\cdot\frac{2-w}{(u^2+v^2+(2-w)^2)^{3/2}}\,dA\\&=\int_{r=0}^2\int_{\theta=0}^{2\pi}\frac{r}{4\pi\epsilon_0}\cdot\frac{2-r}{\big((r\cos\theta)^2+(r\sin\theta)^2+(2-r)^2\big)^{3/2}}\cdot\sqrt{2}\,r\,d\theta\,dr\\&=\int_{r=0}^2\int_{\theta=0}^{2\pi}\frac{1}{4\pi\epsilon_0}\cdot\frac{\sqrt{2}\,r^2(2-r)}{(2r^2+4-4r)^{3/2}}\,d\theta\,dr\\&=\int_{r=0}^2\frac{1}{2\epsilon_0}\cdot\frac{\sqrt{2}\,r^2(2-r)}{2\sqrt{2}(r^2-2r+2)^{3/2}}\,dr=\int_{r=0}^2\frac{1}{4\epsilon_0}\cdot\frac{r^2(2-r)}{((r-1)^2+1)^{3/2}}\,dr.\end{aligned}$$

We make the substitution $\tan\phi=r-1$ so that $\sec\phi=\sqrt{(r-1)^2+1}$ to get

$$\begin{aligned}E_z&=\int_{\phi=-\pi/4}^{\pi/4}\frac{1}{4\epsilon_0}\cdot\frac{(1+\tan\phi)^2(1-\tan\phi)\sec^2\phi\,d\phi}{\sec^3\phi}\\&=\int_{\phi=-\pi/4}^{\pi/4}\frac{1}{4\epsilon_0}\cdot(1+\tan\phi-\tan^2\phi-\tan^3\phi)\cos\phi\,d\phi\end{aligned}$$

By symmetry, since $\cos\phi$ and $\tan^2\phi\cos\phi$ are even and $\tan\phi\cos\phi$ and $\tan^3\phi\cos\phi$ are odd, we have

$$\begin{aligned}E_z&=\frac{1}{2\epsilon_0}\int_{\phi=0}^{\pi/4}(1-\tan^2\phi)\cos\phi\,d\phi=\frac{1}{2\epsilon_0}\int_{\phi=0}^{\pi/4}\left(1-\frac{\sin^2\phi}{\cos^2\phi}\right)\cos\phi\,d\phi\\&=\frac{1}{2\epsilon_0}\int_{\pi=0}^{\pi/4}\left(\frac{\cos^2\phi-\sin^2\phi}{\cos^2\phi}\right)\cos\phi\,d\phi=\frac{1}{2\epsilon_0}\int_{\phi=0}^{\pi/4}\left(\frac{1-2\sin^2\phi}{1-\sin^2\phi}\right)\cos\phi\,d\phi.\end{aligned}$$

We make the substitution $u=\sin\phi$, $du=\cos\phi\,d\phi$ to get

$$\begin{aligned}E_z&=\frac{1}{2\epsilon_0}\int_{u=0}^{1/\sqrt{2}}\frac{1-2u^2}{1-u^2}\,du=\frac{1}{2\epsilon_0}\int_{u=0}^{1/\sqrt{2}}2-\frac{\frac12}{1+u}-\frac{\frac12}{1-u}\,du\\&=\frac{1}{2\epsilon_0}\Big[2u-\tfrac12\ln\tfrac{1+u}{1-u}\Big]_{u=0}^{1/\sqrt{2}}=\frac{1}{2\epsilon_0}\Big(\sqrt{2}-\tfrac12\ln\tfrac{\sqrt{2}+1}{\sqrt{2}-1}\Big)=\frac{1}{2\epsilon_0}\Big(\sqrt{2}-\tfrac12\ln(\sqrt{2}+1)^2\Big)\\&=\frac{1}{2\epsilon_0}\big(\sqrt{2}-\ln(\sqrt{2}+1)\big).\end{aligned}$$

Thus $E(0,0,2)=\frac{1}{2\epsilon_0}\big(\sqrt{2}-\ln(\sqrt{2}+1)\big)(0,0,1)$.

5: Recall that for a scalar-valued function $f: U \subseteq \mathbf{R}^3 \to \mathbf{R}$ the Laplacian of f is given by $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$. For a vector-valued function $F: U \subseteq \mathbf{R}^3 \to \mathbf{R}^3$, given by $F = (P, Q, R)$, we define the **Laplacian** of F to be

$$\nabla^2 F = \left(\nabla^2 P, \nabla^2 Q, \nabla^2 R\right).$$

(a) Show that for $F: U \subseteq \mathbf{R}^3 \to \mathbf{R}^3$ we have $\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \nabla^2 F$.

Solution: When $F = (P, Q, R)$ we have

$$\begin{aligned}
\nabla \times (\nabla \times F) &= \nabla \times \left(\tfrac{\partial R}{\partial y} - \tfrac{\partial Q}{\partial z}\,,\, \tfrac{\partial P}{\partial z} - \tfrac{\partial R}{\partial x}\,,\, \tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\right) \\
&= \left(\tfrac{\partial}{\partial y}\big(\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\big) - \tfrac{\partial}{\partial z}\big(\tfrac{\partial P}{\partial z} - \tfrac{\partial R}{\partial x}\big)\,,\, \cdots\right) \\
&= \left(\tfrac{\partial^2 Q}{\partial x \partial y} - \tfrac{\partial^2 P}{\partial y^2} - \tfrac{\partial^2 P}{\partial z^2} + \tfrac{\partial^2 R}{\partial x \partial z}\,,\, \cdots\right)
\end{aligned}$$

and

$$\begin{aligned}
\nabla\big(\nabla \cdot F\big) - \nabla^2 F &= \nabla\big(\tfrac{\partial P}{\partial x} + \tfrac{\partial Q}{\partial y} + \tfrac{\partial R}{\partial z}\big) - \big(\nabla^2 P, \nabla^2 Q, \nabla^2 R\big) \\
&= \big(\tfrac{\partial^2 P}{\partial x^2} + \tfrac{\partial^2 Q}{\partial x \partial y} + \tfrac{\partial^2 R}{\partial x \partial z}\,,\, \cdots\big) - \big(\tfrac{\partial^2 P}{\partial x^2} + \tfrac{\partial^2 P}{\partial y^2} + \tfrac{\partial^2 P}{\partial z^2}\,,\, \cdots\big) \\
&= \big(\tfrac{\partial^2 Q}{\partial x \partial y} + \tfrac{\partial^2 R}{\partial x \partial z} - \tfrac{\partial^2 P}{\partial y^2} - \tfrac{\partial^2 P}{\partial z^2}\,,\, \cdots\big).
\end{aligned}$$

This shows that the x-components of $\nabla \times (\nabla \times F)$ and $\nabla\big(\nabla \cdot F\big) - \nabla^2 F$ are equal, and similar calculations show that the y- and z- components are also equal.

(b) Show that in a vacuum (where $\rho = 0$ and $J = 0$) the electric and magnetic fields E and B both satisfy the **wave equation**

$$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \quad \text{and} \quad \nabla^2 B = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}.$$

(We remark that it follows from this that $\mu_0 \epsilon_0 = \frac{1}{c^2}$ where c is the speed of light).

Solution: Using the formula from Part (a) together with Maxwell's Equations $\nabla \cdot E = 0$, $\nabla \times E = -\frac{\partial B}{\partial t}$, $\nabla \cdot B = 0$ and $\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$ we obtain

$$\begin{aligned}
\nabla^2 E &= \nabla(\nabla \cdot E) - \nabla \times (\nabla \times E) = \nabla(0) - \nabla \times \big(-\tfrac{\partial B}{\partial t}\big) \\
&= \tfrac{\partial}{\partial t}(\nabla \times B) = \tfrac{\partial}{\partial t}\big(\mu_0 \epsilon_0 \tfrac{\partial E}{\partial t}\big) = \mu_0 \epsilon_0 \tfrac{\partial^2 E}{\partial t^2}
\end{aligned}$$

and similarly

$$\begin{aligned}
\nabla^2 B &= \nabla(\nabla \cdot B) - \nabla \times (\nabla \times B) = \nabla(0) - \nabla \times \big(\mu_0 \epsilon_0 \tfrac{\partial E}{\partial t}\big) \\
&= -\mu_0 \epsilon_0 \tfrac{\partial}{\partial t}(\nabla \times E) = -\mu_0 \epsilon_0 \tfrac{\partial}{\partial t}\big(-\tfrac{\partial B}{\partial t}\big) = \mu_0 \epsilon_0 \tfrac{\partial^2 B}{\partial t^2}
\end{aligned}$$

6: Find a formula for the gradient of a scalar-valued function in spherical coordinates.

Solution: In case there is not enough time to present this material in the lectures, we provide a solution which may be applied to other orthogonal coordinate systems. Let $(x,y,z) = g(u,v,w)$ be a change of coordinate map. Let g_u, g_v and g_w denote the columns of the derivative matrix Dg so we have $Dg = (g_u, g_v, g_w)$. Suppose that $\{g_u, g_v, g_w\}$ is an orthogonal set. Let $e_u = \frac{g_u}{|g_u|}$, $e_v = \frac{g_v}{|g_v|}$ and $e_w = \frac{g_w}{|g_w|}$. Note that $\{e_u, e_v, e_w\}$ is an orthonormal set so we have $(e_u, e_v, e_w)^T(e_u, e_v, e_w) = I$. Let $f : U \subseteq \mathbf{R}^3 \to \mathbf{R}$ be a smooth scalar-valued function. When we use the map g to change coordinates, we replace $f(x,y,z)$ by the map $h(u,v,w) = f(g(u,v,w))$. By the Chain Rule, we have $Dh = Df\, Dg$ and so

$$\begin{aligned} Df &= Dh \cdot (Dg)^{-1} = Dh \cdot (g_u, g_v, g_w)^{-1} \\ &= Dh \cdot \left((e_u, e_v, e_w) \begin{pmatrix} |g_u| & & \\ & |g_v| & \\ & & |g_w| \end{pmatrix} \right)^{-1} = Dh \begin{pmatrix} \frac{1}{|g_u|} & & \\ & \frac{1}{|g_v|} & \\ & & \frac{1}{|g_w|} \end{pmatrix} \begin{pmatrix} {e_u}^T \\ {e_v}^T \\ {e_w}^T \end{pmatrix} \end{aligned}$$

and so

$$\begin{aligned} \nabla f = (Df)^T &= \big(e_u, e_v, e_w\big) \begin{pmatrix} \frac{1}{|g_u|} & & \\ & \frac{1}{|g_v|} & \\ & & \frac{1}{|g_w|} \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \\ \frac{\partial h}{\partial w} \end{pmatrix} \\ &= \frac{1}{|g_u|} \frac{\partial h}{\partial u}\, e_u + \frac{1}{|g_v|} \frac{\partial h}{\partial v}\, e_v + \frac{1}{|g_w|} \frac{\partial h}{\partial w} e_w. \end{aligned}$$

In particular, when g is the spherical coordinates map given by

$$(x,y,z) = g(r,\phi,\theta) = \big(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi\big)$$

we have

$$Dg = \begin{pmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{pmatrix}$$

so that $|g_r| = 1$, $|g_\phi| = r$ and $|g_\theta| = r\sin\phi$, and so

$$\nabla f = \frac{\partial h}{\partial r} e_r + \frac{1}{r}\frac{\partial h}{\partial \phi} e_\phi + \frac{1}{r\sin\phi}\frac{\partial h}{\partial \theta} e_\theta.$$