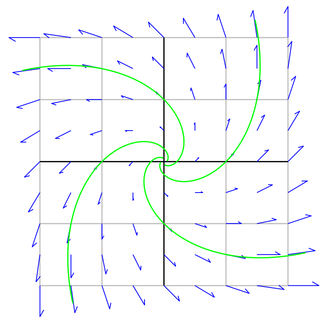
ECE 206 Advanced Calculus 2, Solutions to Assignment 5

1: Let $F(x,y) = \big(P(x,y), Q(x,y)\big) = (x-y, x+y)$.

(a) Sketch the vector field $\frac{1}{4}F$, along with some of its flow lines.

Solution: A plot is shown below. We remark that although you are not asked to find equations for the flow lines, in polar coordinates they are given by $r = ae^{\theta}$.



(b) Verify directly that $\int_C F \cdot T \, dL = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ when $D = \{(x,y)|x^2+y^2 \leq 1\}$ and C is the circle given by $(x,y) = (\cos t, \sin t)$.

Solution: Let $\alpha(t) = (\cos t, \sin t)$. Then

$$\int_C F \cdot T \, dL = \int_{t=0}^{2\pi} F\big(\alpha(t)\big) \cdot \alpha'(t) \, dt = \int_{t=0}^{2\pi} \big(\cos t - \sin t, \cos t + \sin t\big) \cdot \big(-\sin t, \cos t\big) \, dt$$
$$= \int_0^{2\pi} \big(-\sin t \cos t + \cos^2 t + \sin^2 t + \sin t \cos t\big) \, dt = \int_0^{2\pi} 1 \, dt = 2\pi$$

On the other hand,

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 1 + 1 \, dA = 2 \cdot \text{Area}(D) = 2\pi.$$

2: Find the work done by the force $F(x,y,z) = \big(2x+yz\,,\, 2y+xz\,,\, 2z+xy\big)$ acting on an object which moves along the curve $(x,y,z) = \alpha(t) = \left(\frac{2+t^2}{1+t}\,,\, \frac{2+t^3}{1+t^2}\,,\, \frac{2+t^4}{1+t^3}\right)$ with $0 \leq t \leq 2$.

Solution: Notice that $\nabla \times F = (x-x, y-y, z-z) = (0,0,0)$ and so we might be able to find a scalar potential for F. By inspection, we find that for $g(x,y,z) = x^2+y^2+z^2+xyz$ we have $F = \nabla g$. By the Conservative Field Theorem, the work is

$$W = \int_\alpha F \cdot T \, dL = \int_\alpha \nabla g \cdot T \, dL = g\big(\alpha(2)\big) - g\big(\alpha(0)\big)$$
$$= g(3,2,2) - g(2,2,2) = 29 - 20 = 9.$$

3: Find the flux of the vector field $F(x,y,z) = \left(x^2 + \sqrt{yz}\,,\, y + x\sqrt{z}\,,\, z + x\sqrt{y}\right)$ across the boundary surface of the region $D = \left\{(x,y,z)\middle| x^2 \le z \le 2 - x^2\,,\, 0 \le y \le z\right\}$.

Solution: Let S be the boundary surface. By the Divergence Theorem (Gauss' Theorem), the flux is

$$\begin{aligned}\Phi &= \iint_S F \cdot N\ dA = \iiint_D \nabla \cdot F\ dV = \int_{x=-1}^{1} \int_{z=x^2}^{2-x^2} \int_{y=0}^{z} (2x+1+1) dy\, dz\, dx \\ &= \int_{x=-1}^{1} \int_{z=x^2}^{2-x^2} 2(x+1)z\ dz\, dx = \int_{x=-1}^{1} (x+1)\big((2-x^2)^2 - (x^2)^2\big)\, dx \\ &= \int_{x=-1}^{1} (x+1)(4-4x^2)\, dx = \int_{x=-1}^{1} 4 + 4x - 4x^2 - 4x^3\ dx \\ &= 8 + 0 - \tfrac{8}{3} - 0 = \tfrac{16}{3}.\end{aligned}$$

4: Find the work done by the force $F = (y^2 z, x+z, x^2 + yz)$ acting on an object which moves counterclockwise (looking down from above) once around the boundary of $S = \left\{(x,y,z)\middle| 0 \le x \le 1\,,\, 1-x \le y \le 1+x\,,\, z = x^2\right\}$.

Solution: Let C be the boundary curve. By Stokes' Theorem, the work is

$$W = \int_C F \cdot T\ dL = \iint_S (\nabla \times F) \cdot N\ dA.$$

Note that $\nabla \times F = \left(z - 1, y^2 - 2x, 1 - 2yz\right)$. We can parametrize S by

$$(x,y,z) = \sigma(s,t) = \left(s, t, s^2\right) \ \text{ with } \ 0 \le s \le 1\,,\, 1 - s \le t \le 1 + s$$

and then the normal vector is given by

$$\sigma_s \times \sigma_t = (1,0,2s) \times (0,1,0) = (-2s, 0, 1).$$

Note that the normal vector point upwards (as desired). Thus the work is

$$\begin{aligned}W &= \iint_S (\nabla \times F) \cdot N\ dA = \int_{s=0}^{1} \int_{t=1-s}^{1+s} \left(s^2 - 1, t^2 - 2s, 1 - 2s^2 t\right) \cdot (-2s, 0, 1)\ dt\, ds \\ &= \int_{s=0}^{1} \int_{t=1-s}^{1+s} -2s^3 + 2s + 1 - 2s^2 t\ dt\, ds = \int_{s=0}^{1} \Big[(1 + 2s - 2s^3)\, t - s^2 t^2\Big]_{t=1-s}^{1+s} ds \\ &= \int_{s=0}^{1} (1 + 2s - 2s^3)(2s) - (s^2)(4s)\ ds = \int_{s=0}^{1} 2s + 4s^2 - 4s^3 - 4s^4\ ds \\ &= 1 + \tfrac{4}{3} - 1 - \tfrac{4}{5} = \tfrac{8}{15}.\end{aligned}$$

5: Find the flux of the vector field F given by $F(x,y,z)=(xz,-yz,1+y^2)$ across the surface S which given by $S=\{(x,y,z)\big|z=\cos^{-1}(x^2+y^2)\,,\,x^2+y^2\leq 1\}$ with outwards pointing normal vector.

Solution: Let C be the boundary curve of S, which is given by $(x,y,z)=\alpha(t)=(\cos t,\sin t,0)$. Note that $\nabla \cdot F = z-z=0$ and so we might be able to find a vector potential for F. By inspection, we note that for $G(x,y,z)=(-y,xy^2,xyz)$ we have $\nabla\times G=F$. By Stoke's Theorem, the flux is

$$\begin{aligned}\Phi &= \iint_S F\cdot N\,dA = \iint_S (\nabla\times G)\cdot N\,dA = \int_C G\cdot T\,dL\\ &= \int_{t=0}^{2\pi} \big(-\sin t,\cos t\sin^2 t,0\big)\cdot(-\sin t,\cos t,0)\,dt\\ &= \int_{t=0}^{2\pi} \sin^2 t+\sin^2 t\cos^2 t\,dt = \int_{t=0}^{2\pi}\sin^2 t+\tfrac14\sin^2(2t)\,dt\\ &= \pi+\tfrac14\,\pi=\tfrac{5\pi}{4}.\end{aligned}$$

6: Let $F(x,y)=\left(\frac{-y}{x^2+y^2}\,,\,\frac{x}{x^2+y^2}\right)$.

(a) Show that when C is the curve given by $(x,y)=\alpha(t)=\big(r(t)\cos\theta(t)\,,\,r(t)\sin\theta(t)\big)$ for $a\leq t\leq b$, where $r(t)$ and $\theta(t)$ are smooth with $r(t)>0$, we have $\displaystyle\int_C F\cdot T\,dL=\theta(b)-\theta(a)$.

Solution: We have

$$\begin{aligned}\int_C F\cdot T\,dL &= \int_{t=a}^{b} F\big(\alpha(t)\big)\cdot\alpha'(t)\,dt\\ &= \int_{t=a}^{b}\Big(-\frac{r\sin\theta}{r^2},\frac{r\cos\theta}{r^2}\Big)\cdot\Big(r'\cos\theta-r\sin\theta\cdot\theta'\,,\,r'\sin\theta+r\cos\theta\cdot\theta'\Big)\,dt\\ &= \int_{t=a}^{b} -\frac{r'\sin\theta\cos\theta}{r}+\sin^2\theta\cdot\theta'+\frac{r'\sin\theta\cos\theta}{r}+\cos^2\theta\cdot\theta'\,dt\\ &= \int_{t=a}^{b}\theta'(t)\,dt=\theta(b)=\theta(a).\end{aligned}$$

(b) Find $\displaystyle\int_C F\cdot T\,dL$ when C is given by $(x,y)=\alpha(t)=(3-t^2,t)$ for $1\leq t\leq 2$.

Solution: By Part (a), the value of $\displaystyle\int_C F\cdot T\,dL$ is equal to the change in the angle $\Delta\theta=\theta(2)-\theta(1)$ as we move from $\alpha(1)=(2,1)$ along the curve (which is moving counterclockwise around the origin) to $\alpha(2)=(-1,2)$. Since $(2,1)$ and $(-1,2)$ are orthogonal,

$$\int_C F\cdot T\,dL=\Delta\theta=\tfrac{\pi}{2}.$$

To be very explicit, we have $\alpha(t)=\big(r(t)\cos\theta(t),r(t)\sin\theta(t)\big)$ with $r(t)=\sqrt{x^2+y^2}=\sqrt{(3-t^2)^2+t^2}$ and $\theta(t)=\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)=\cos^{-1}\left(\frac{3-t^2}{\sqrt{(3-t^2)^2+t^2}}\right)$ and so

$$\int_C F\cdot T\,dL=\theta(2)-\theta(1)=\cos^{-1}\big(\tfrac{-1}{\sqrt5}\big)-\cos^{-1}\big(\tfrac{2}{\sqrt5}\big)=\tfrac{\pi}{2}$$