ECE 206 Advanced Calculus 2, Solutions to Assignment 4

1: A cord, carrying an unevenly distributed charge, is wound around the cone $z = \sqrt{x^2+y^2}$ following the curve $(x,y,z) = \alpha(t) = (t\cos t, t\sin t, t)$ with $0 \le t \le 4$. The charge density (charge per unit length) of the cord at position (x,y,z) is given by $f(x,y,z) = z$. Find the total charge of the cord.

Solution: We have $\alpha'(t) = \big(\cos t - t\sin t\,,\, \sin t + t\cos t\,,\, 1\big)$ so that

$$\left|\alpha'(t)\right|^2 = (\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t) + (\sin^2 t - 2t\sin t\cos t + t^2\cos^2 t) + 1 = 2 + t^2.$$

Using the substitution $u = 2+t^2$ so that $du = 2t\,dt$, the total charge on the cord is

$$\begin{aligned} Q &= \int_{t=0}^{4} f\big(\alpha(t)\big)\left|\alpha'(t)\right| dt = \int_{t=0}^{4} t\sqrt{2+t^2}\,dt = \int_{u=2}^{18} \tfrac{1}{2}u^{1/2}\,du = \Big[\tfrac{1}{3}u^{3/2}\Big]_{u=2}^{18} \\ &= \tfrac{1}{3}(18\sqrt{18} - 2\sqrt{2}) = \tfrac{1}{3}(54\sqrt{2} - 2\sqrt{2}) = \tfrac{52\sqrt{2}}{3}. \end{aligned}$$

2: A long evenly charged wire lies along the z-axis. The electric field in the region surrounding the wire is given by $E = 2kq\Big(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, 0\Big)$, where q is the charge density (charge per unit length) of the wire and k is a constant (which depends on the units used). A small object of unit charge moves along the curve $(x,y,z) = \alpha(t) = \big(1-t, 2t, 1+3t\big)$ for $0 \le t \le 1$. Find the work done by the electric field on the object.

Solution: We have $\alpha'(t) = (-1,2,3)$ and $E\big(\alpha(t)\big) = 2kq\Big(\frac{1-t}{1-2t+5t^2}\,,\, \frac{2t}{1-2t+5t^2}\,,\, 0\Big)$. Using the substitution $u = 1-2t+5t^2$ so that $du = (-2+10t)dt$, we find that the total work done is

$$\begin{aligned} W &= \int_\alpha F\cdot T\;dL = \int_{t=0}^{1} 2kq\,\frac{-(1-t)+2(2t)}{1-2t+5t^2}\;dt = 2kq\int_{t=0}^{1}\frac{-1+5t}{1-2t+5t^2}\;dt \\ &= kq\int_{u=1}^{4}\frac{du}{u} = kq\Big[\ln u\Big]_{u=1}^{4} = kq\ln 4. \end{aligned}$$

3: A gas expands and rotates with velocity field $V(x,y,z) = (x-y, x+y, z)$. Find the rate (volume per unit time) at which the gas passes through the triangle with vertices at $(1,0,-1)$, $(1,3,2)$ and $(0,1,2)$.

Solution: The top view of the triangle (that is the projection of the triangle to the xy-plane) is the triangle with vertices $(1,0)$, $(1,3)$ and $(0,1)$, which is given by $0 \le x \le 1$, $1-x \le y \le 1+2x$. The plane through the points $a = (1,0,-1)$, $b = (1,3,2)$ and $c = (0,1,2)$ has direction vectors $u = (b-a) = (0,3,3)$ and $v = c-a = (-1,1,3)$, and so it has normal vector $n = \frac{1}{3}u\times v = (0,1,1)\times(-1,1,3) = (2,-1,1)$, and so its equation is of the form $2x - y + z = d$ for some constant d. We put in $(x,y,z) = (1,0,-1)$ to get $d=1$, so the equation of the plane is $2x-y+z=1$, or equivalently $z = 1-2x+y$. Thus the given triangle is given by $0\le x\le 1$, $1-x\le y\le 1+2x$ and $z = 1-2x+y$. We parametrize the triangle by

$$(x,y,x) = \sigma(s,t) = (s,t,1-2s+t)\quad\text{, with }\; 0\le s\le 1\,,\; 1-s\le t\le 1+2s.$$

We have

$$\sigma_s\times\sigma_t = \begin{pmatrix}1\\0\\-2\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\1\end{pmatrix}\quad\text{and}\quad F\big(\sigma(s,t)\big) = \begin{pmatrix}s-t\\s+t\\1-2s+t\end{pmatrix}$$

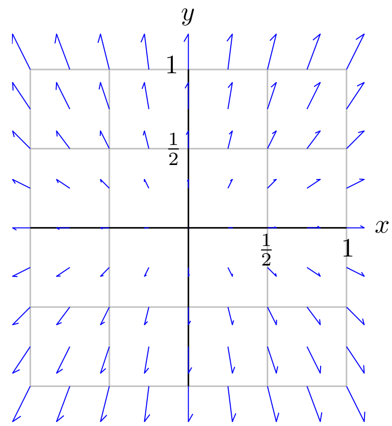
and so the rate at which the gas passes through the triangle is

$$\begin{aligned} \Phi &= \int_\sigma F\cdot N\,dA = \int_{s=0}^{1}\int_{t=1-s}^{t=1+2s} 2(s-t)-(s+t)+(1-2s+t)\;dt\,ds \\ &= \int_{s=0}^{1}\int_{t=1-s}^{1+2s} 1-s-2t\;dt\,ds = \int_{s=0}^{1}\Big[(1-s)t - t^2\Big]_{t=1-s}^{1+2s} ds \\ &= \int_{s=0}^{1}(1-s)(1+2s)-(1+2s)^2-(1-s)(1-s)+(1-s)^2\;ds \\ &= \int_{s=0}^{1}(1+s-2s^2)-(1+4s+4s^2)-(1-2s+s^2)+(1-2s+s^2)\;ds \\ &= \int_{s=0}^{1}-3s-6s^2\;ds = \Big[-\tfrac{3}{2}s^2-2s^3\Big]_{s=0}^{1} = -\tfrac{3}{2}-2 = -\tfrac{7}{2}. \end{aligned}$$

4: Let $F(x,y)=(x,2y)$.

(a) Make an accurate sketch of the vector field $\frac{1}{4}F$.

Solution: At each point (x,y) we draw the vector $\frac{1}{4}F(x,y)=\left(\frac{x}{4},\frac{y}{2}\right)$.



(b) Find the equations of the flow lines of F.

Solution: We need to solve the DE $y'=\dfrac{2y}{x}$. This is separable. We can write it as $\frac{1}{y}\,dy=\frac{2}{x}\,dx$ and integrate to get $\ln|y|=2\ln|x|+c$, or equivalently $y=Ae^{2\ln x}=Ax^2$, so the integral curves are parabolas.

5: The surface obtained by revolving the circle $(x-1)^2+z^2=1$ in the xz-plane about the z-axis can be given parametrically by

$$(x,y,z)=\sigma(\theta,\phi)=\Big((1+\cos\phi)\cos\theta\,,\,(1+\cos\phi)\sin\theta\,,\,\sin\phi\Big).$$

with $0\le\theta\le 2\pi$ and $0\le\phi\le 2\pi$. Find the mass of this surface given that its density (mass per unit area) at position (x,y,z) is given by $f(x,y,z)=1+z^2$.

Solution: We have

$$\sigma_\theta\times\sigma_\phi=\begin{pmatrix}-(1+\cos\phi)\sin\theta\\-(1+\cos\phi)\cos\theta\\0\end{pmatrix}\times\begin{pmatrix}-\sin\phi\cos\theta\\-\sin\phi\sin\theta\\\cos\phi\end{pmatrix}=\begin{pmatrix}(1+\cos\phi)\cos\phi\cos\theta\\(1+\cos\phi)\cos\phi\sin\theta\\(1+\cos\phi)\sin\phi\end{pmatrix}$$

and so

$$\big|\sigma_\phi\times\sigma_\theta\big|=(1+\cos\phi)\sqrt{\cos^2\phi\cos^2\theta+\cos^2\phi\sin^2\theta+\sin^2\phi}=(1+\cos\phi)\sqrt{\cos^2\phi+\sin^2\phi}=1+\cos\phi$$

and we have

$$f\big(\sigma(\phi,\theta)\big)=1+\sin^2\phi$$

so the mass of the surface is

$$M=\int_\sigma f\,dA=\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}f\big(\sigma(\phi,\theta)\big)\big|\sigma_\phi\times\sigma_\theta\big|d\theta\,d\phi=\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}(1+\sin^2\phi)(1+\cos\phi)\,d\theta\,d\phi$$
$$=2\pi\int_{\phi=0}^{\pi}1+\cos\phi+\sin^2\!\phi+\sin^2\!\phi\cos\phi\;d\phi=2\pi\big(\pi+0+\tfrac{\pi}{2}+0\big)=3\pi^2$$

since $\displaystyle\int_0^\pi 1\,d\phi=\pi$, $\displaystyle\int_0^\pi\cos\phi\;d\phi=\Big[\sin\phi\Big]_0^\pi=0$, $\displaystyle\int_0^\pi\sin^2\!\phi\;d\phi=\int_0^\pi\tfrac{1}{2}(1-\cos 2\phi)\,d\phi=\Big[\tfrac{1}{2}\phi-\tfrac{1}{4}\sin 2\phi\Big]_0^\pi=\tfrac{\pi}{2}$ and $\displaystyle\int_0^\pi\sin^2\!\phi\cos\phi\;d\phi=\Big[\tfrac{1}{3}\sin^3\phi\Big]_0^\pi=0$.

6: Let $F(x,y,z) = (xz, yz, x^2+y^2)$. Find the flux of F across the boundary surface of the solid given by $x^2+y^2 \le z \le 1$.

Solution: The solid is bounded above by the disc given by $z=1$ with $x^2+y^2 \le 1$, and it is bounded below by the portion of the paraboloid given by $z = x^2+y^2$ with $x^2+y^2 \le 1$. We parametrize the disc by

$$(x,y,z) = \sigma(r,\theta) = \big(r\cos\theta\,,\, r\sin\theta\,,\, 1\big) \text{ with } 0 \le r \le 1\,,\, 0 \le \theta \le 2\pi.$$

We have

$$\sigma_r \times \sigma_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \quad \text{and} \quad F\big(\sigma(r,\theta)\big) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r^2 \end{pmatrix}.$$

Note that the normal vector $(0,0,r)$ points upwards, out from the solid, so the outward flux across the top disc is

$$\Phi_{\text{top}} = \int_\sigma F \cdot N \, dA = \int_{r=0}^1 \int_{\theta=0}^{2\pi} r^3 \, d\theta\, dr = 2\pi \int_{r=0}^1 r^3\, dr = 2\pi \Big[\tfrac{1}{4}r^4\Big]_0^1 = \tfrac{\pi}{2}\,.$$

We parametrize the lower parabolic surface by

$$(x,y,z) = \sigma(r,\theta) = \big(r\cos\theta\,,\, r\sin\theta\,,\, r^2\big) \text{ with } 0 \le r \le 1\,,\, 0 \le \theta \le 2\pi.$$

We have

$$\sigma_r \times \sigma_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2r \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix} \quad \text{and} \quad F\big(\sigma(r,\theta)\big) = \begin{pmatrix} r^3\cos\theta \\ r^3\sin\theta \\ r^2 \end{pmatrix}.$$

Note that the normal vector $(-2r^2\cos\theta, -2r^2\sin\theta, r)$ is pointing upwards into the solid, so the outward flux across this lower paraboloid is

$$\begin{aligned}\Phi_{\text{bot}} &= -\int_\sigma F \cdot N\, dA = -\int_{r=0}^1 \int_{\theta=0}^{2\pi} -2r^5\cos^2\theta - 2r^5\sin^2\theta + r^3 \, d\theta\, dr = \int_{r=0}^1 \int_{\theta=0}^{2\pi} 2r^5 - r^3\, d\theta\, dr \\ &= 2\pi \int_{r=0}^1 2r^5 - r^3\, dr = 2\pi\Big[\tfrac{1}{3}r^6 - \tfrac{1}{4}r^4\Big]_{r=0}^1 = 2\pi\big(\tfrac{1}{3} - \tfrac{1}{4}\big) = \tfrac{\pi}{6}.\end{aligned}$$

Thus the total flux across the boundary surface is

$$\Phi = \Phi_{\text{top}} + \Phi_{\text{bot}} = \tfrac{\pi}{2} + \tfrac{\pi}{6} = \tfrac{2\pi}{3}.$$