

ECE 206 Advanced Calculus 2, Solutions to Assignment 3

- 1:** Find the volume of the region which lies under the surface $z = e^{x+y}$ and above the triangle in the xy -plane with vertices at $(0, 0)$, $(1, 1)$ and $(0, 2)$.

Solution: The triangle is the set $D = \{(x, y) | 0 \leq x \leq 1, x \leq y \leq 2 - x\}$, so the volume is

$$\begin{aligned} V &= \int_D e^{x+y} dV = \int_{x=0}^1 \int_{y=x}^{2-x} e^{x+y} dy dx = \int_{x=0}^1 \left[e^{x+y} \right]_{y=x}^{2-x} dx \\ &= \int_0^1 e^2 - e^{2x} dx = \left[e^2 x - \frac{1}{2} e^{2x} \right]_0^1 = e^2 - \frac{1}{2} e^2 + \frac{1}{2} = \frac{1}{2} (e^2 + 1). \end{aligned}$$

- 2:** A flat plate is in the shape of the region D bounded by $x + y = -1$ and $x + y^2 = 1$. The charge density (charge per unit area) of the plate at position (x, y) is given by $f(x, y) = xy$. Find the total charge of the plate.

Solution: Note that $D = \{(x, y) | -1 \leq y \leq 2, -1 - y \leq x \leq 1 - y^2\}$ and so the charge is

$$\begin{aligned} Q &= \int_D f dA = \int_{y=-1}^2 \int_{x=-1-y}^{1-y^2} xy dx dy = \int_{y=-1}^2 \left[\frac{1}{2} x^2 y \right]_{x=-1-y}^{1-y^2} dy = \int_{-1}^2 \frac{1}{2} ((1 - y^2)^2 - (1 + y)^2) y dy \\ &= \int_{y=-1}^2 \frac{1}{2} (1 - 2y^2 + y^4 - 1 - 2y - y^2) y dy = \int_{y=-1}^2 \frac{1}{2} (y^4 - 3y^2 - 2y) y dy \\ &= \int_{y=-1}^2 \left(\frac{1}{2} y^5 - \frac{3}{2} y^3 - y^2 \right) dy = \left[\frac{1}{12} y^6 - \frac{3}{8} y^4 - \frac{1}{3} y^3 \right]_{-1}^2 \\ &= \left(\frac{16}{3} - 6 - \frac{8}{3} \right) - \left(\frac{1}{12} - \frac{3}{8} + \frac{1}{3} \right) = -\frac{10}{3} - \frac{1}{24} = -\frac{81}{24} = -\frac{27}{8}. \end{aligned}$$

- 3:** Find the mass of the solid tetrahedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ with density given by $f(x, y, z) = 1/(1 + x)$.

Solution: The tetrahedron is the set $D = \{(x, y, z) | 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}$, and so the mass is

$$\begin{aligned} M &= \int_D f dV = \int_{x=0}^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} \frac{1}{1+x} dz dy dx = \int_{x=0}^1 \int_{y=0}^{1-x} \left[\frac{z}{1+x} \right]_{z=0}^{1-x-y} dy dx \\ &= \int_{x=0}^1 \int_{y=0}^{1-x} \frac{1-x-y}{1+x} dy dx = \int_{x=0}^1 \left[\frac{(1-x)y - \frac{1}{2}y^2}{1+x} \right]_{y=0}^{1-x} dx = \frac{1}{2} \int_{x=0}^1 \frac{(1-x)^2}{1+x} dx \\ &= \frac{1}{2} \int_0^1 \frac{x^2 - 2x + 1}{x + 1} dx = \frac{1}{2} \int_0^1 \left(x - 3 + \frac{4}{x+1} \right) dx \\ &= \frac{1}{2} \left[\frac{1}{2} x^2 - 3x + 4 \ln(x+1) \right]_0^1 = \frac{1}{2} \left(\frac{1}{2} - 3 + 4 \ln 2 \right) = 2 \ln 2 - \frac{5}{4}. \end{aligned}$$

4: Find $\int_D x^2 + y \, dV$ where D is the tetrahedron bounded by $x + y + z = 2$, $z = 2$, $x = 1$ and $y = x$.

Solution: Although not completely necessary, it helps to sketch the tetrahedron, and to do this it helps to find the vertices. The vertices are at $(1, -1, 2)$, $(0, 0, 2)$, $(1, 1, 0)$ and $(1, 1, 2)$ (each vertex can be found by solving three of the four equations $x + y + z = 2$, $z = 2$, $x = 1$ and $y = x$). The region D can be described in several ways, for example we have

$$\begin{aligned} D &= \{(x, y, z) | 0 \leq x \leq 1, -x \leq y \leq x, 2 - x - y \leq z \leq 2\} \\ &= \{(x, y, z) | 0 \leq x \leq 1, 2 - 2x \leq z \leq 2, 2 - x - z \leq y \leq x\} \\ &= \{(x, y, z) | 0 \leq z \leq 2, 1 - \frac{1}{2}z \leq x \leq 1, 2 - x - z \leq y \leq x\}. \end{aligned}$$

Using the first of the above three descriptions of D we have

$$\begin{aligned} \int_D x^2 + y \, dV &= \int_{x=0}^1 \int_{y=-x}^x \int_{z=2-x-y}^2 x^2 + y \, dz \, dy \, dx = \int_{x=0}^1 \int_{y=-x}^x [(x^2 + y)z]_{z=2-x-y}^2 dy \, dx \\ &= \int_{x=0}^1 \int_{y=-x}^x (x^2 + y)(2 - (2 - x - y)) dy \, dx = \int_{x=0}^1 \int_{y=-x}^x (x^2 + y)(x + y) dy \, dx \\ &= \int_{x=0}^1 \int_{y=-x}^x x^3 + (x^2 + x)y + y^2 dy \, dx = \int_{x=0}^1 \left[x^3 y + \frac{1}{2}(x^2 + x)y^2 + \frac{1}{3}y^3 \right]_{y=-x}^x dx \\ &= \int_{x=0}^1 2x^4 + \frac{2}{3}x^3 dx = \left[\frac{2}{5}x^5 + \frac{1}{6}x^4 \right]_0^1 = \frac{2}{5} + \frac{1}{6} = \frac{17}{30}. \end{aligned}$$

5: Use the change of coordinates map $(x, y) = g(u, v) = (1 + 2u + v, 1 + u + 3v)$ to find $\int_D e^{x-y} \, dx \, dy$ where D is the parallelogram with vertices at $(1, 1)$, $(3, 2)$, $(4, 5)$ and $(2, 4)$.

Solution: In matrix form, the change of coordinated map g is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = g(u, v) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and we have} \quad Dg(u, v) = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Note that g sends the square $C = \{(u, v) | 0 \leq u \leq 1, 0 \leq v \leq 1\}$ to the given parallelogram D . Let $f(x, y) = e^{x-y}$. Then we have

$$f(g(u, v)) = f(1 + 2u + v, 1 + u + 3v) = e^{(1+2u+v)-(1+u+3v)} = e^{u-2v}.$$

and so

$$\begin{aligned} \iint_D e^{x-y} \, dx \, dy &= \iint_C f(g(u, v)) |Dg(u, v)| \, du \, dv = \int_{u=0}^1 \int_{v=0}^1 e^{u-2v} \cdot 5 \, dv \, du = 5 \int_{u=0}^1 \left[-\frac{1}{2}e^{u-2v} \right]_{v=0}^1 du \\ &= -\frac{5}{2} \int_0^1 e^{u-2} - e^u \, du = \frac{5}{2} [e^u - e^{u-2}]_0^1 = \frac{5}{2} (e - e^{-1} - 1 + e^{-2}). \end{aligned}$$

We remark that this problem could also be solved, without a change of coordinates, by cutting the given parallelogram D into the three regions

$$\begin{aligned} D &= \{(x, y) | 1 \leq x \leq 2, \frac{x+1}{2} \leq y \leq 3x - 2\} \cup \{(x, y) | 2 \leq x \leq 3, \frac{x+1}{2} \leq y \leq \frac{x+6}{2}\} \\ &\quad \cup \{(x, y) | 3 \leq x \leq 4, 3x - 7 \leq y \leq \frac{x+6}{2}\}. \end{aligned}$$

- 6: Use polar coordinates to find the volume of the region which lies outside the cylinder $x^2 + y^2 = 1$, inside the cylinder $x^2 + y^2 = 2x$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

Solution: The region $D = \{(x, y) | x^2 + y^2 \geq 1, x^2 + y^2 \leq 2x\}$ lies in the right half of the plane and can be expressed conveniently using polar coordinates. Write $x = r \cos \theta$ and $y = r \sin \theta$ with $r > 0$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $x + 2 + y^2 \geq 1 \iff r^2 \geq 1 \iff r \geq 1$ and $x^2 + y^2 \leq 2x \iff r^2 \leq 2r \cos \theta \iff r \leq 2 \cos \theta$ and we have $1 = r = 2 \cos \theta$ when $\theta = \pm \frac{\pi}{3}$. Thus the polar coordinates map $(x, y) = g(r, \theta) = (r \cos \theta, r \sin \theta)$ sends the region $C = \{(r, \theta) | -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}, 1 \leq r \leq 2 \cos \theta\}$ to the region D . The required volume is

$$\begin{aligned} V &= 2 \iint_D \sqrt{4 - x^2 - y^2} \, dx \, dy = 2 \iint_C \sqrt{4 - r^2} \cdot r \, dr \, d\theta = 4 \int_{\theta=0}^{\pi/3} \int_{r=1}^{2 \cos \theta} r \sqrt{4 - r^2} \, dr \, d\theta \\ &= 4 \int_{\theta=0}^{\pi/3} \left[-\frac{1}{3}(4 - r^2)^{3/2} \right]_{r=1}^{2 \cos \theta} d\theta = \int_{\theta=0}^{\pi/3} \frac{4}{3} (3^{3/2} - (4 - 4 \cos^2 \theta)^{3/2}) \, d\theta \\ &= \int_{\theta=0}^{\pi/3} \frac{4}{3} (3\sqrt{3} - 8 \sin^3 \theta) \, d\theta = \int_{\theta=0}^{\pi/3} 4\sqrt{3} - \frac{32}{3} (1 - \cos^2 \theta) \sin \theta \, d\theta \\ &= \left[4\sqrt{3} \theta + \frac{32}{3} \cos \theta - \frac{32}{9} \cos^3 \theta \right]_{\theta=0}^{\pi/3} = \left(\frac{4\sqrt{3}\pi}{3} + \frac{16}{3} - \frac{4}{9} \right) - \left(\frac{32}{3} - \frac{32}{9} \right) = \frac{4\sqrt{3}\pi}{3} - \frac{20}{9}. \end{aligned}$$

- 7: Use spherical coordinates to find the mass of the solid in the shape of the region D given by $z \geq \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 \leq 4$ with density given by $f(x, y, z) = 2 - z$.

Solution: Let $g : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be the spherical coordinates map given by

$$(x, y, z) = g(r, \phi, \theta) = (r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi).$$

Recall that the Jacobian of this map is $|Dg(r, \phi, \theta)| = r^2 \sin \phi$. The given solid lies above the xy -plane, and for $r > 0$, $0 \leq \phi < \frac{\pi}{2}$ and $0 \leq \theta \leq 2\pi$ we have

$$\begin{aligned} z \geq \sqrt{x^2 + y^2} &\iff r \cos \phi \geq r \sin \phi \iff \tan \phi \leq 1 \iff 0 \leq \phi \leq \frac{\pi}{4}, \\ x^2 + y^2 + z^2 \leq 4 &\iff r^2 \leq 4 \iff 0 \leq r \leq 2, \end{aligned}$$

and so the map g sends the region $C = \{(r, \phi, \theta) | 0 \leq r \leq 2, 0 \leq \phi \leq \frac{\pi}{4}, 0 \leq \theta \leq 2\pi\}$ to the desired region $D = \{(x, y, z) | z \geq \sqrt{x^2 + y^2}, x^2 + y^2 + z^2 \leq 4\}$. We have $f(g(r, \phi, \theta)) = 2 - r \cos \phi$, and so the mass is

$$\begin{aligned} M &= \int_D f(x, y, z) \, dV = \iiint_C f(g(r, \phi, \theta)) |Dg(r, \phi, \theta)| \, dr \, d\phi \, d\theta \\ &= \int_{r=0}^2 \int_{\phi=0}^{\pi/4} \int_{\theta=0}^{2\pi} (2 - r \cos \phi) \cdot r^2 \sin \phi \, d\theta \, d\phi \, dr = 2\pi \int_{r=0}^2 \int_{\phi=0}^{\pi/4} 2r^2 \sin \phi - r^3 \cos \phi \sin \phi \, d\phi \, dr \\ &= 2\pi \int_{r=0}^2 \left[-2r^2 \cos \phi - \frac{1}{2} r^3 \sin^2 \phi \right]_{\phi=0}^{\pi/4} = 2\pi \int_0^2 -\sqrt{2} r^2 - \frac{1}{4} r^3 + 2r^2 \, dr \\ &= 2\pi \left[\frac{1}{3} (2 - \sqrt{2}) r^3 - \frac{1}{16} r^4 \right]_0^2 = 2\pi \left(\frac{8}{3} (2 - \sqrt{2}) - 1 \right) = \frac{2\pi}{3} (13 - 8\sqrt{2}). \end{aligned}$$