ECE 206 Advanced Calculus 2, Solutions to Assignment 2

1: Let $(u,v)=f(t)=\big(\cos t+2, 2\sin t-1\big)$ and let $(x,y)=g(u,v)=\left(\frac{u}{v},\frac{v}{u}\right)$. Use the Chain Rule to find the tangent vector to the curve $r(t)=g(f(t))$ at the point where $t=\frac{\pi}{2}$.

Solution: We express the solution in two ways; with and without matrix notation. First we express the solution without matrix notation. We use the Chain Rule in the form

$$\frac{dx}{dt}=\frac{\partial x}{\partial u}\frac{du}{dt}+\frac{\partial x}{\partial v}\frac{dv}{dt}$$
$$\frac{dy}{dt}=\frac{\partial y}{\partial u}\frac{du}{dt}+\frac{\partial y}{\partial v}\frac{dv}{dt}$$

We have

$$\frac{\partial x}{\partial u}=\frac{1}{v}\,,\ \frac{\partial x}{\partial v}=-\frac{u}{v^2}\,,\ \frac{\partial y}{\partial u}=-\frac{v}{u^2}\,,\ \frac{\partial y}{\partial v}=\frac{1}{u}\,,\ \frac{du}{dt}=-\sin t\,,\text{ and }\frac{dv}{dt}=2\cos t\,.$$

When $t=\frac{\pi}{2}$ we have $u=\cos t+2=2$ and $v=2\sin t-1=1$, and so

$$\frac{\partial x}{\partial u}=1\,,\ \frac{\partial x}{\partial v}=-2\,,\ \frac{\partial y}{\partial u}=-\frac{1}{4}\,,\ \frac{\partial y}{\partial v}=\frac{1}{2}\,,\ \frac{du}{dt}=-1\,,\text{ and }\frac{dv}{dt}=0\,.$$

Put all these values into the two formulas given by the Chain Rule to get

$$\frac{dx}{dt}=\frac{\partial x}{\partial u}\frac{du}{dt}+\frac{\partial x}{\partial v}\frac{dv}{dt}=(1)(-1)+(-2)(0)=-1$$
$$\frac{dy}{dt}=\frac{\partial y}{\partial u}\frac{du}{dt}+\frac{\partial y}{\partial v}\frac{dv}{dt}=\big(-\tfrac{1}{4}\big)(-1)+\big(\tfrac{1}{2}\big)(0)=\tfrac{1}{4}\,.$$

Thus the tangent vector is $r'\left(\frac{\pi}{4}\right)=\left(x'\left(\frac{\pi}{4}\right),y'\left(\frac{\pi}{4}\right)\right)=\left(-1,\frac{1}{4}\right)$.

Here is the same solution in matrix notation. By the Chain Rule, we have $r'(t)=Dg(f(t))\,f'(t)$, where

$$r'(t)=\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{pmatrix}\,,\ f'(t)=\begin{pmatrix}\frac{du}{dt}\\ \frac{dv}{dt}\end{pmatrix}=\begin{pmatrix}-\sin t\\ 2\cos t\end{pmatrix}\,,\text{ and }Dg(u,v)=\begin{pmatrix}\frac{\partial x}{\partial u}&\frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}\end{pmatrix}=\begin{pmatrix}\frac{1}{v}&-\frac{u}{v^2}\\ -\frac{v}{u^2}&\frac{1}{u}\end{pmatrix}$$

When $t=\frac{\pi}{2}$ we have $(u,v)=f\left(\frac{\pi}{2}\right)=(2,1)$, and $f'\left(\frac{\pi}{2}\right)=\begin{pmatrix}-1\\0\end{pmatrix}$ and $Dg(2,1)=\begin{pmatrix}1&-2\\-\frac14&\frac12\end{pmatrix}$ so the tangent vector at $t=\frac{\pi}{2}$ is

$$r'\left(\tfrac{\pi}{2}\right)=Dg(2,1)\ f'\left(\tfrac{\pi}{2}\right)=\begin{pmatrix}1&-2\\-\frac14&\frac12\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix}=\begin{pmatrix}-1\\ \frac14\end{pmatrix}$$

2: Let $u=f(x,y,z)=4\,x\tan^{-1}\left(\frac{y}{z}\right)$ where $(x,y,z)=g(s,t)=\left(s^3+t,\sqrt{s}\,t,\frac{t}{s}\right)$. Use the Chain Rule to find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$ when $(s,t)=(1,-2)$.

Solution: First we give a solution which does not use matrix notation. Note that when $(s,t)=(1,-2)$ we have $(x,y,z)=(-1,-2,-2)$, and at this point we have

$$\frac{\partial u}{\partial x}=4\tan^{-1}\frac{y}{z}=\pi\,,\ \frac{\partial u}{\partial y}=\frac{4x}{1+(y/z)^2}\cdot\frac{1}{z}=1\,,\ \frac{\partial u}{\partial z}=\frac{4x}{1+\left(\frac{y}{z}\right)^2}\left(-\frac{y}{z^2}\right)=-1$$

$$\frac{\partial x}{\partial s}=3s^2=3\,,\ \frac{\partial x}{\partial t}=1\,,\ \frac{\partial y}{\partial s}=\frac{t}{2\sqrt{s}}=-1\,,\ \frac{\partial y}{\partial t}=\sqrt{s}=1\,,\ \frac{\partial z}{\partial s}=-\frac{t}{s^2}=2\,,\ \frac{\partial z}{\partial t}=\frac{1}{s}=1$$

and so

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial s}=(\pi)(3)+(1)(-1)+(-1)(2)=3\pi-3\ \text{, and}$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial t}=(\pi)(1)+(1)(1)+(-1)(1)=\pi$$

Here is the same solution, using matrix notation. Write $u=h(s,t)=f\big(g(s,t)\big)$. By the Chain Rule, we have $Dh(s,t)=Df\big(g(s,t)\big)Dg(s,t)$. When $(s,t)=(1,-2)$ we have $(x,y,z)=h(s,t)=(-1,-2,-2)$, and at this point

$$\begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial u}{\partial t}\end{pmatrix}=Dh(s,t)=Df(x,y,z)\cdot Dg(s,t)=\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t}\end{pmatrix}$$

$$=\left(4\tan^{-1}\frac{y}{z}\ ,\ \frac{4x(1/z)}{1+(y/z)^2}\ ,\ \frac{4x(-y/z^2)}{1+(y/z)^2}\right)\begin{pmatrix}3s^2 & 1\\ \frac{1}{2\sqrt{t}} & \frac{1}{2\sqrt{s}}\\ -\frac{t}{s^2} & \frac{1}{s}\end{pmatrix}$$

$$=\big(\pi\,,\,1\,,\,-1\big)\begin{pmatrix}3 & 1\\ -1 & 1\\ 2 & 1\end{pmatrix}=\big(3\pi-3\,,\,\pi\big).$$

3: Let $u = f(x,y,z) = (x+y)e^{y^2+z}$.

(a) Find $\nabla f(1,2,-4)$.

Solution: $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = e^{y^2+z}\big(1, 1+(x+y)(2y), (x+y)\big)$, so $\nabla f(1,2,-4) = (1,13,3)$.

(b) Find the equation of the tangent plane at $(1,2,-4)$ to the surface $f(x,y,z) = 3$.

Solution: The gradient $(1,13,3)$ is a normal vector, so the equation is of the form $x + 13y + 3z = c$, and by putting in $(x,y,z) = (1,2,-4)$, we find that $c = 15$. Thus the equation is $x + 13y + 3z = 15$.

(c) Find $D_u f(1,2,-4)$, where $u = \frac{1}{7}(2,-3,6)$.

Solution: $D_u f(1,2,-4) = \nabla f(1,2,-4) \cdot u = \frac{1}{7}(1,13,3) \cdot (2,-3,6) = -\frac{19}{7}$.

4: Let $f(x,y) = x^2y - y^3$. Find $\nabla f(3,-1)$, then for each of the values $m = 0, 6, 6\sqrt{2}$ and 10, find a unit vector u, if one exists, such that $D_u f(3,-1) = m$.

Solution: $\nabla f(x,y) = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right) = (2xy\,,\, x^2 - 3y^2)$ and so $\nabla f(3,-1) = (6,6)$. For each value of m, we need to find a vector $u = (a,b)$ with $a^2+b^2 = 1$ such that $m = D_u f(3,-1) = \nabla f(3,-1) \cdot u = (6,6) \cdot (a,b) = 6a+6b$, thus we need to solve the two equations $a^2 + b^2 = 1 \;\; (1)$ and $a + b = \frac{1}{6}m \;\; (2)$.

When $m = 0$, equation (2) becomes $a + b = 0$ so $b = -a$. Put $b = -a$ into equation (1) to get $a^2 + (-a)^2 = 1 \Longrightarrow 2a^2 = 1 \Longrightarrow a^2 = \frac{1}{2} \Longrightarrow a = \pm\frac{\sqrt{2}}{2}$. Since $b = -a$, we obtain $(a,b) = \pm\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

When $m = 6$, equation (2) becomes $a+b = 1$ so $b = 1-a$. Put this into equation (1) to get $a^2+(1-a)^2 = 1 \Longrightarrow a^2 + 1 - 2a + a^2 = 1 \Longrightarrow 2a^2 - 2a = 0 \Longrightarrow 2a(a-1) = 0 \Longrightarrow a = 0$ or $a = 1$. Since $b = 1-a$, we obtain $(a,b) = (0,1)$ or $(1,0)$.

When $m = 6\sqrt{2}$, equation (2) becomes $a + b = \sqrt{2}$ so $b = \sqrt{2} - a$. Put this into equation (1) to get $a^2 + (\sqrt{2}-a)^2 = 1 \Longrightarrow a^2 + 2 - 2\sqrt{2}a + a^2 = 1 \Longrightarrow 2a^2 - 2\sqrt{2}a + 1 = 0 \Longrightarrow 2\big(a - \frac{1}{\sqrt{2}}\big)^2 = 0 \Longrightarrow a = \frac{1}{\sqrt{2}}$. Since $b = \sqrt{2} - a$, we obtain $(a,b) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Finally, note that since the vector $(a,b) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is in the direction of the gradient vector $(6,6)$, it gives the maximum possible value for the directional derivative. So the maximum possible value for $D_u f(3,-1)$ is equal to $6\sqrt{2}$; there is no unit vector such that $D_u f(3,-1) = 10$.

5: A boy is standing at the point $(5,10,2)$ on a hill whose shape is given by

$$z = \frac{600}{100 + 4x^2 + y^2}$$

(where x, y and z are in meters).

(a) At the point where the boy is standing, in which direction is the slope steepest?

Solution: Write $z = f(x,y) = \frac{600}{100+4x^2+y^2}$ and $a = (5,10)$. Then $\nabla f = \left(\frac{-4800\,x}{(100+4x^2+y^2)^2}, \frac{-1200\,y}{(100+4x^2+y^2)^2}\right)$ and so $\nabla f(a) = \left(-\frac{24{,}000}{90{,}000}, -\frac{12{,}000}{90{,}000}\right) = \left(-\frac{4}{15}, -\frac{2}{15}\right) = \frac{2}{15}(-2,-1)$. Thus the slope is the steepest in the direction of the unit vector $\frac{1}{\sqrt{5}}(-2,-1)$.

(b) If the boy walks southeast, then will he be ascending or descending?

Solution: The southeasterly direction is in the direction of the unit vector $v = \frac{1}{\sqrt{2}}(1,-1)$, and we have $D_v f(a) = \frac{2}{15\sqrt{2}}(-2,-1) \cdot (1,-1) = -\frac{\sqrt{2}}{15} < 0$, so the boy would be descending.

(c) If the boy walks in the direction of steepest slope, then at what angle (from the horizontal) will he be climbing?

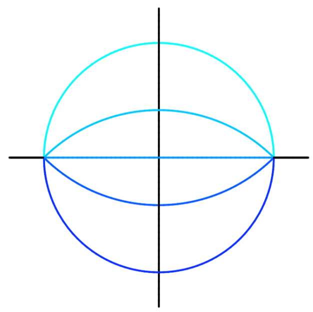
Solution: If the boy walks in the direction of the unit vector $u = \frac{1}{|\nabla f(a)|}\nabla f(a)$, then the slope in that direction is $D_u f(a) = |\nabla f| = \frac{2}{15}\big|(-2,-1)\big| = \frac{2\sqrt{5}}{15}$, so the angle of ascent is $\theta = \tan^{-1}\frac{2\sqrt{5}}{15} \cong 16.6°$.

6: The temperature around the outer circle of a metal disc of radius 1 meter is held constant, with the top half of the circle held at 0° C and the bottom half of the circle held at 20° C.
It can be shown that the temperature at all points of the disc is given by

$$T(x,y) = 10 + \frac{20}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right).$$

(a) Sketch the isotherms (level curves of constant temperature) $T = 0, 5, 10, 15, 20$.

Solution: We know that $T = 0$ along the top half of the boundary circle, and $T = 20$ along the bottom half. We have $T = 5 \iff \frac{20}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = -5 \iff \tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = -\frac{\pi}{4} \iff \frac{2y}{x^2+y^2-1} = -1 \iff 2y = -x^2-y^2+1 \iff x^2+y^2+2y = 1 \iff x^2+(y+1)^2 = 2$, and this is the circle of radius $\sqrt{2}$ centered at $(0,-1)$. Similarly, we have $T = 15 \iff x^2+y^2-2y = 1 \iff x^2+(y-1)^2 = 2$, and this is the circle of radius $\sqrt{2}$ centered at $(0,1)$. Also, we have $T = 10 \iff \frac{20}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = 0 \iff \frac{2t}{x^2+y^2-1} = 0 \iff 2y = 0 \iff y = 0$. These isotherms are shown below.



(b) Find $T\left(\frac{1}{2},\frac{1}{2}\right)$ and $\nabla T\left(\frac{1}{2},\frac{1}{2}\right)$.

Solution: We have $T\left(\frac{1}{2},\frac{1}{2}\right) = 10 + \frac{20}{\pi}\tan^{-1}\left(\frac{1}{-1/2}\right) = 10 - \frac{20}{\pi}\tan^{-1}2 \cong 2.95$. Also, we have

$$\begin{aligned}\nabla T(x,y) = \left(\frac{\partial T}{\partial x},\frac{\partial T}{\partial y}\right) &= \frac{20}{\pi}\left(\frac{1}{1+\left(\frac{2y}{x^2+y^2-1}\right)^2}\cdot\frac{-4xy}{(x^2+y^2-1)^2},\ \frac{1}{1+\left(\frac{2y}{x^2+y^2-1}\right)^2}\cdot\frac{2(x^2+y^2-1)-4y^2}{(x^2+y^2-1)^2}\right)\\ &= \frac{20}{\pi}\left(\frac{-4xy}{(x^2+y^2-1)^2+4y^2},\ \frac{2(x^2-y^2-1)}{(x^2+y^2-1)^2+4y^2}\right)\end{aligned}$$

and so $\nabla T\left(\frac{1}{2},\frac{1}{2}\right) = \frac{20}{\pi}\left(\frac{-1}{\frac{1}{4}+1},\frac{-2}{\frac{1}{4}+1}\right) = \frac{20}{\pi}\left(-\frac{4}{5},-\frac{8}{5}\right) = \frac{16}{\pi}(-1,-2)$.

(c) Find the equation of the tangent line to the isotherm through $\left(\frac{1}{2},\frac{1}{2}\right)$.

Solution: The tangent line to this isotherm is perpendicular to $\nabla T\left(\frac{1}{2},\frac{1}{2}\right) = \frac{16}{\pi}(-1,-2)$, so it is in the direction of the vector $(2,-1)$. A vector equation for the tangent line is $(x,y) = \left(\frac{1}{2},\frac{1}{2}\right) + t\,(2,-1)$.

(d) Show that if an ant starts at the point $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ (where the temperature is 0°) and it walks on the disc in the direction of ∇T (that is, in the direction in which the temperature increases most rapidly), then it will walk along the circle of radius $\sqrt{3}$ centered at $(2,0)$.

Solution: By the formula for $\nabla T(x,y)$ that we found in part (b), we see that at each point $\nabla T(x,y)$ is in the direction of the vector $\big(-4xy, 2(x^2-y^2-1)\big)$, which has slope $\frac{2(x^2-y^2-1)}{-4xy} = \frac{-x^2+y^2+1}{2xy}$. The circle of radius $\sqrt{3}$ centered at $(2,0)$ has equation $(x-2)^2+y^2 = 3$. Differentiating implicitly gives $2(x-2)+2y\,y' = 0$ and so $y' = \frac{2-x}{y}$. For any point (x,y) on this circle, we have $(x-2)^2+y^2 = 3 \Longrightarrow x^2-4x+4+y^2 = 3 \Longrightarrow 4x = x^2+y^2+1$, and so at such a point $y' = \frac{2-x}{y} = \frac{4x-2x^2}{2xy} = \frac{(x^2+y^2+1)-2x^2}{2xy} = \frac{-x^2+y^2+1}{2xy}$, which is the same as the slope of the gradient vector.