ECE 206 Advanced Calculus 2, Solutions to Assignment 12

Extra problems, not to be handed in

1: Find $\displaystyle\int_0^{2\pi}\frac{dt}{(5+3\cos t)^2}$.

Solution: Write $I=\displaystyle\int_0^{2\pi}\frac{dt}{(5+3\cos t)^2}$. Let $\alpha(t)=e^{i\,t}$ for $0\le t\le 2\pi$. Then

$$\begin{aligned}I&=\int_\alpha\frac{-\frac{i}{z}\,dz}{\left(5+\frac32\left(z+\frac1z\right)\right)^2}=\int_\alpha\frac{-4i\,z\,dz}{(10z+3(z^2+1))^2}=\int_\alpha\frac{-4i\,z\,dz}{(3z^2+10z+3)^2}=\int_\alpha\frac{-4i\,z\,dz}{9\left(z+3\right)^2\left(z+\frac13\right)^2}\\&=2\pi\,i\,\eta\big(\alpha,-\tfrac13\big)F'\left(-\tfrac13\right)=2\pi\,i\,F'\left(-\tfrac13\right)\end{aligned}$$

where $F(z)=\dfrac{-4\,z}{9(z+3)^2}$. We have $F'(z)=-\dfrac{4i}{9}\cdot\dfrac{(z+3)^2-2z(z+3)}{(z+3)^4}=\dfrac{4i}{9}\cdot\dfrac{z-3}{(z+3)^3}$, and so

$$I=2\pi\,i\cdot\frac{4i}{9}\cdot\frac{-\frac{10}{3}}{\left(\frac83\right)^3}=2\pi\cdot\tfrac49\cdot\tfrac{10}{3}\cdot\tfrac{3^3}{8^3}=\tfrac{5\pi}{32}\,.$$

2: Find $\displaystyle\int_0^\infty\frac{x^2\,dx}{x^4+x^2+1}$.

Solution: Write $I=\displaystyle\int_0^\infty\frac{x^2\,dx}{x^4+x^2+1}$. Let $f(z)=\dfrac{z^2}{z^4+z^2+1}$. Note that

$$z^4+z^2+10\iff z^2=\tfrac{-1\pm\sqrt3\,i}{2}=e^{\pm i\,2\pi/3}\iff z=\pm e^{\pm i\,\pi/3}\iff z=\pm e^{i\,\pi/3},\pm e^{i\,2\pi/3}\,.$$

Let α be the loop that follows first the line λ, given by $\lambda(t)=t$ for $-R\le t\le R$, and then the semicircle σ given by $\sigma(t)=Re^{i\,t}$ for $0\le t\le\pi$, so we have

$$\int_\alpha f=\int_\lambda f+\int_\sigma f\,.$$

Let α_1 be the loop around the right half of α and let α_2 be the loop around the left half of α so that α_1 winds once around $e^{i\,\pi/3}$, and α_2 winds once around $e^{i\,2\pi/3}$, and we have

$$\int_\alpha f=\int_{\alpha_1}f+\int_{\alpha_2}f\,.$$

Let

$$F(z)=\frac{z^2}{\left(z+e^{i\,\pi/3}\right)\left(z-e^{i\,2\pi/3}\right)\left(z+e^{i\,2\pi/3}\right)}\text{ and }G(z)=\frac{z^2}{\left(z-e^{i\,\pi/3}\right)\left(z+e^{i\,\pi/3}\right)\left(z+e^{i\,2\pi/3}\right)}\,.$$

Then we have

$$\begin{aligned}\int_\alpha f&=\int_{\alpha_1}f+\int_{\alpha_2}f=\int_{\alpha_1}\frac{F(z)\,dz}{z-e^{i\,\pi/3}}+\int_{\alpha_2}\frac{G(z)}{z-e^{i\,2\pi/3}}\\&=2\pi\,i\,F\big(e^{i\,\pi/3}\big)+2\pi\,i\,G\big(e^{i\,2\pi/3}\big)\\&=2\pi\,i\cdot\frac{e^{i\,2\pi/3}}{(2e^{i\,\pi/3})(1)(\sqrt3\,i)}+2\pi\,i\cdot\frac{e^{i\,4\pi/3}}{(-1)(\sqrt3\,i)(2e^{i\,2\pi/3})}\\&=\tfrac{\pi}{\sqrt3}\left(e^{i\,\pi/3}-e^{i\,2\pi/3}\right)=\tfrac{\pi}{\sqrt3}\,.\end{aligned}$$

Also, we have

$$\int_\lambda f=\int_{-R}^R\frac{t^2\;dt}{t^4+t^2+1}=2\int_0^R\frac{t^2\;dt}{t^4+t^2+1}\longrightarrow 2I\text{ as }R\to\infty$$

and

$$\left|\int_\sigma f\right|=\left|\int_0^{2\pi}\frac{(Re^{i\,t})^2(i\,Re^{i\,t})\;dt}{(Re^{i\,t})^4+(Re^{i\,t})^2+1}\right|\le 2\pi\cdot\frac{R^3}{R^4-(R^2+1)}\longrightarrow 0\text{ as }R\to\infty\,.$$

It follows that $2I=\frac{\pi}{\sqrt3}$ and so $I=\frac{\pi}{2\sqrt3}$.

3: Find $\displaystyle\int_0^\infty \frac{x\,\sin x}{x^4+4}\,dx$.

Solution: Write $I=\displaystyle\int_0^\infty \frac{x\,\sin x}{x^4+4}\,dx$. Let $f(z)=\dfrac{z\,e^{i\,z}}{z^4+4}$. Note that

$$z^4=0\iff z^4=-4=4e^{i\pi}\iff z=\pm\sqrt{2}\,e^{\pm i\,\pi/4}\iff z=\pm 1\pm i\,.$$

Let α be the loop which follows first the line λ given by $\lambda(t)=t$ for $-R\leq t\leq R$, then the semicircle σ given by $\sigma(t)=R\,e^{i\,t}$ for $0\leq t\leq\pi$ so that

$$\int_\alpha f=\int_\lambda f+\int_\sigma f\,.$$

Let α_1 be the loop around the right half of α and let α_2 be the loop around the left half of α so that α_1 winds once around $1+i$ and α_2 winds once around $-1+i$ and we have

$$\int_\alpha f=\int_{\alpha_1} f+\int_{\alpha_2} f\,.$$

Also, let

$$\begin{aligned}F(z)&=\frac{z\,e^{i\,z}}{(z+(1+i))(z-(-1+i))(z+(-1+i))}\ \text{, and}\\ G(z)&=\frac{z\,e^{i\,z}}{(z-(1+i))(z+(1+i))(z+(-1+i))}\,.\end{aligned}$$

Then we have

$$\begin{aligned}\int_\alpha f=\int_{\alpha_1} f+\int_{\alpha_2 f}&=\int_{\alpha_1}\frac{F(z)\,dz}{z-(1+i)}+\int_{\alpha_2}\frac{G(z)\,dz}{z-(-1+i)}\\ &=2\pi\,i\,\eta(\alpha_1,1+i)\,F(1+i)+2\pi\,i\,\eta(\alpha_2,-1+i)\,G(-1+i)\\ &=2\pi\,i\cdot 1\cdot\frac{(1+i)e^{-1+i}}{(2(i+1))(2)(2i)}+2\pi\,i\cdot 1\cdot\frac{(-1+i)e^{-1-i}}{(-2)(2i)(2(-1+i))}\\ &=\tfrac{\pi}{4}\big(e^{-1+i}-e^{-1-i}\big)=\tfrac{\pi}{4}\big(2\,i\,e^{-1}\sin 1\big)=i\,\tfrac{\pi\sin 1}{2e}\,.\end{aligned}$$

Also

$$\int_\lambda f=\int_{-R}^R\frac{t\,e^{it}}{t^4+4}\,dt=\int_{-R}^R\frac{t\cos t}{t^4+4}\,dt+i\int_{-R}^R\frac{t\sin t}{t^4+4}\,dt\longrightarrow 0+i\,2I\ \text{as}\ R\to\infty$$

and since $\left|e^{iRe^{it}}\right|=\left|e^{i(R\cos t+i\,R\sin t)}\right|=\left|e^{-R\sin t+i\,R\cos t}\right|=e^{-R\sin t}\leq 1$ for $0\leq t\leq\pi$ we have

$$\left|\int_\sigma f\right|=\left|\int_0^\pi\frac{\left(Re^{it}\right)\left(e^{iRe^{it}}\right)\left(iRe^{it}\right)}{\left(Re^{it}\right)^4+4}\,dt\right|\leq\frac{\pi\,R^2}{R^4-4}\longrightarrow 0\ \text{as}\ R\to\infty\,.$$

It follows that $2I=\frac{\pi\sin 1}{2e}$ so $I=\frac{\pi\sin 1}{4e}$.

4: Find $\displaystyle\int_0^\infty \left(\frac{\ln x}{1+x^2}\right)^2 dx$.

Solution: Write $I = \displaystyle\int_0^\infty \left(\frac{\ln x}{1+x^2}\right)^2 dx$. Let $f(z) = \left(\dfrac{\log z}{1+z^2}\right)^2$ where $\log z$ denotes the branch of the logarithm given by $\log z = \ln|z| + i\,\theta(z)$ with $-\frac{\pi}{2} \leq \theta(z) \leq \frac{3\pi}{2}$. Let α be the loop which follows the line λ given by $\lambda(t) = t$ for $\epsilon \leq t \leq R$, then the semicircle σ given by $\sigma(t) = R\,e^{i\,t}$ for $0 \leq t \leq \pi$, then the line segment μ^{-1} where μ is given by $\mu(t) = -t$ for $0 \leq t \leq \pi$, and then the semicircle τ^{-1} where τ is given by $\tau(t) = \epsilon\,e^{i\,t}$ for $0 \leq t \leq \pi$, so we have

$$\int_\alpha f = \int_\lambda f + \int_\sigma f - \int_\mu f - \int_\tau f\,.$$

Let $F(z) = \left(\dfrac{\log z}{z+i}\right)^2$. Note that $F'(z) = 2 \cdot \dfrac{\log z}{z+i} \cdot \dfrac{\frac{1}{z}(z+i) - \log z}{(z+i)^2}$. We have

$$\begin{aligned}\int_\alpha f &= \int_\alpha \frac{F(z)\,dz}{(z-i)^2} = 2\pi\,i\,\eta(\alpha,i)\,F'(i) = 2\pi\,i \cdot 1 \cdot 2 \cdot \frac{i\,\frac{\pi}{2}}{2i} \cdot \frac{(-i)(2i) - i\,\frac{\pi}{2}}{(2i)^2} \\ &= \tfrac{\pi^2}{4}\left(2i - \tfrac{\pi}{2}\right) = -\tfrac{\pi^3}{8} - i\,\tfrac{\pi^2}{2}\,.\end{aligned}$$

Also, we have

$$\int_\lambda f = \int_\epsilon^R \left(\frac{\ln t}{1+t^2}\right)^2 dt \longrightarrow I \text{ as } \epsilon \to 0 \text{ and } R \to \infty\,,$$

$$\left|\int_\sigma f\right| = \left|\int_0^\pi \left(\frac{\ln R + i\,t}{1+(R\,e^{i\,t})^2}\right)^2 \left(i\,R\,e^{i\,t}\right)\,dt\right| \leq \pi\left(\frac{\ln R + \pi}{R^2-1}\right)^2 R \longrightarrow 0 \text{ as } R \to \infty\,,$$

$$\left|\int_\tau f\right| = \left|\int_0^\pi \left(\frac{\ln \epsilon + i\,t}{1+(\epsilon\,e^{i\,t})^2}\right)^2 \left(i\,\epsilon\,e^{i\,t}\right)\,dt\right| \leq \pi\left(\frac{\ln \epsilon + \pi}{1-\epsilon^2}\right)^2 \epsilon \longrightarrow 0 \text{ as } \epsilon \to 0\,,$$

and

$$\begin{aligned}\int_\mu f &= \int_\epsilon^R \left(\frac{\ln t + i\,\pi}{1+t^2}\right)^2 (-1)\,dt = -\int_\epsilon^R \frac{\left((\ln t)^2 - \pi^2\right) + i\,2\pi\ln t}{(1+t^2)^2}\,dt \\ &= -\int_\epsilon^R \left(\frac{\ln t}{1+t^2}\right)^2 dt + \int_\epsilon^R \frac{\pi^2}{(1+t^2)^2}\,dt - i\int_\epsilon^R \frac{2\pi\ln t}{(1+t^2)^2}\,dt \\ &\longrightarrow -I + J - i\,K\,,\end{aligned}$$

where $J = \displaystyle\int_0^\infty \frac{\pi^2\,dt}{(1+t^2)^2}$ and $K = \displaystyle\int_0^\infty \frac{2\pi\ln t}{(1+t^2)^2}\,dt$. Making the substitution $\tan\theta = t$, $\sec\theta = \sqrt{1+t^2}$, $\sec^2\theta\,d\theta = dt$ gives

$$J = \int_0^{\pi/2} \frac{\pi^2 \sec^2\theta\,d\theta}{\sec^4\theta} = \int_0^{\pi/2} \pi^2\cos^2\theta\;d\theta = \tfrac{\pi^3}{4}$$

and so

$$\int_\mu f \longrightarrow -I + \tfrac{\pi^3}{4} - i\,K\,.$$

It follows that $2I - \frac{\pi^3}{4} + i\,K = -\frac{\pi^3}{8} - i\,\frac{\pi^2}{2}$ and so we have

$$I = \tfrac{1}{2}\left(\tfrac{\pi^3}{4} - \tfrac{\pi^3}{8}\right) = \tfrac{\pi^3}{16}\,.$$

Incidentally, we also find that $K = -\frac{\pi^2}{2}$.

5: For a function $f(x)$ with $x \in \mathbf{R}$, the **Fourier transform** of $f(x)$ is defined to be the function $F(\omega) = \mathcal{F}(f)(\omega)$ defined for $s \in \mathbf{R}$ by

$$F(\omega) = \mathcal{F}(f)(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\,\omega x}\, dx\,.$$

Given $F(\omega)$ the function $f(x)$ can be recovered using the inverse transform

$$f(x) = \mathcal{F}^{-1}(F)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\,\omega x}\, d\omega.$$

Find the Fourier transform of $f(x) = \dfrac{1}{1+x^2}$.

Solution: The Fourier transform is

$$F(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{1+x^2}\, dx = \int_{-\infty}^{\infty} g_\omega(t)\, dt$$

where $g_\omega(z) = \dfrac{e^{-i\omega z}}{z^2+1}$ for $z \in \mathbf{C}$.

Suppose first that $\omega \le 0$. Let α be the loop which follows first the line λ given by $\lambda(t) = t$ for $-R \le t \le R$, then the semicircle σ given by $\sigma(t) = R\,e^{i\,t}$ for $0 \le t \le \pi$ so that

$$\int_\alpha g_\omega = \int_\lambda g_\omega + \int_\sigma g_\omega\,.$$

Setting $G_\omega(z) = \dfrac{e^{-i\omega z}}{z+i}$ we have $\displaystyle\int_\alpha g_\omega = 2\pi i\, G_\omega(i) = 2\pi i\, \frac{e^\omega}{2i} = \pi e^\omega$. On the other hand, we have

$$\int_\lambda g_\omega = \int_{-R}^{R} g_\omega(t)\, dt \longrightarrow F(\omega) \text{ as } R \to \infty \text{ , and}$$

$$\left|\int_\sigma g_\omega\right| = \left|\int_0^\pi \frac{e^{-i\omega Re^{it}}\, iRe^{it}}{R^2 e^{i2t}+1}\, dt.\right| \le \pi \cdot \frac{R}{R^2-1} \longrightarrow 0 \text{ as } R \to \infty$$

since $\left|e^{-i\omega Re^{it}}\right| = e^{\omega R \sin t} \le 1$ when $0 \le t \le \pi$ (since $\omega < 0$). It follows that $F(\omega) = \pi e^\omega$.

When $\omega \ge 0$, a similar argument, using the loop α which follows the line λ^{-1} where $\lambda(t) = t$ for $-R \le t \le R$ followed by the semicircle σ given by $\sigma(t) = -Re^{it}$ for $0 \le t \le \pi$, and setting $G_\omega(z) = \dfrac{e^{-i\omega z}}{z-i}$ so that $\displaystyle\int_\alpha g_\omega = 2\pi\, i\, G_\omega(-i) = \pi e^{-\omega}$, shows that $F(\omega) = \pi e^{-\omega}$. Thus for all values of ω we have

$$F(\omega) = \pi\, e^{-|\omega|}.$$

6: For a function $f(t)$ defined for $0 < t \in \mathbf{R}$, the **Laplace transform** of $f(t)$ is the function $F(s) = \mathcal{L}(f)(s)$ defined for all s in a set of the form $\{s \in \mathbf{C} | \mathrm{Re}(s) > c\}$ for some $c \in \mathbf{R}$, by

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

Given $F(s)$ the function $f(t)$ can be recovered using the inverse transform

$$f(t) = \mathcal{L}^{-1}(F)(t) = \frac{1}{2\pi\, i}\lim_{R\to\infty}\int_{\lambda_{a,R}} e^{st}F(s)\,ds\,,$$

where $a > c$ and $\lambda_{a,R}(u) = a + i\,u$ for $-R \le u \le R$.

Find the inverse Laplace transform of $F(s) = \dfrac{2s+3}{s^2+4}$ defined for $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 0$.

Solution: The inverse Laplace transform of $F(s)$ is given by

$$f(t) = \frac{1}{2\pi\, i}\lim_{R\to\infty}\int_{\lambda_R} e^{st}\cdot\frac{2s+3}{s^2+4}\,ds = \frac{1}{2\pi\, i}\lim_{R\to\infty}\int_{\lambda_R} g_t(z)\,dz$$

where $g_t(z) = e^{zt}\cdot\dfrac{2z+3}{z^2+4} = \dfrac{e^{zt}(2z+3)}{(z-2i)(z+2i)}$ for $z \in \mathbf{C}$ with $z \ne \pm 2i$ and where λ_R is the line given by $\lambda_R(u) = 1 + iu$ for $-R \le u \le R$.

Let $\alpha = \alpha_R$ be the loop which follows the line $\lambda = \lambda_R$, then the line μ^{-1} where $\mu = \mu_R$ is given by $\mu(u) = u + iR$ for $0 \le u \le 1$, then the semicircle $\sigma = \sigma_R$ given by $\sigma(u) = iRe^{iu}$ for $0 \le u \le \pi$, and then the line $\nu = \nu_R$ given by $\nu(u) = u - iR$ for $0 \le u \le 1$ so that we have

$$\int_\alpha g_t = \int_\lambda g_t - \int_\mu g_t + \int_\sigma g_t + \int_\nu g_t.$$

Let α_1 be the loop around the top half of α and let α_2 be the loop around the bottom half of α (so that $2i$ lies in α_1 and $-2i$ lies in α_2), and let $G_t(z) = \dfrac{e^{zt}(2z+3)}{(z+2i)}$ and $H_t(z) = \dfrac{e^{zt}(2z+3)}{(z-2i)}$ to get

$$\begin{aligned}\int_\alpha g_t &= \int_{\alpha_1} g_t + \int_{\alpha_2} g_t = 2\pi\, i\, G_t(2i) + 2\pi\, i\, H_t(-2i) = 2\pi\, i\left(\frac{e^{i\,2t}(4i+3)}{4i} + \frac{e^{-i\,2t}(-4i+3)}{-4i}\right)\\ &= 2\pi\, i\cdot 2\mathrm{Re}\Big(\frac{e^{i\,2t}(4i+3)}{4i}\Big) = 2\pi\, i\cdot \mathrm{Re}\Big(\frac{(\cos 2t + i\sin 2t)(4i+3)}{2i}\Big) = 2\pi\, i\left(2\cos 2t + \tfrac{3}{2}\sin 2t\right)\end{aligned}$$

On the other hand we have

$$\begin{aligned}&\int_\lambda g_t \longrightarrow 2\pi\, i\, f(t) \text{ as } R\to\infty,\\ &\left|\int_\mu g_t\right| = \left|\int_0^1 e^{(u+iR)t}\cdot\frac{2(u+iR)+3}{(u+iR)^2+4}\,du\right| \le \frac{2(1+R)+3}{(R-1)^2-4} \longrightarrow 0 \text{ as } R\to\infty,\\ &\left|\int_\nu g_t\right| = \left|\int_0^1 e^{(u-iR)t}\cdot\frac{2(u-iR)+3}{(u-iR)^2+4}\,du\right| \le \frac{2(1+R)+3}{(R-1)^2-4} \longrightarrow 0 \text{ as } R\to\infty, \text{ and}\\ &\left|\int_\sigma g_t\right| = \left|\int_0^\pi e^{itRe^{iu}}\cdot\frac{2iRe^{iu}+3}{(iRe^{iu})^2+4}\cdot(-Re^{iu})\,du\right| \le \int_0^\pi e^{-tR\sin u}\cdot\frac{2R^2+3R}{R^2-4}\,du\\ &\qquad = \frac{2R^2+R}{R^2-4}\cdot 2\int_0^{\pi/2} e^{-tR\sin u}\,du \le \frac{4R^2+2R}{R^2-4}\int_0^{\pi/2} e^{-2tRu/\pi}\,du\\ &\qquad = \frac{4R^2+2R}{R^2-4}\cdot\frac{-\pi}{2tR}\Big[e^{-2tRu/\pi}\Big]_0^{\pi/2} = \frac{4R^2+2R}{R^2-4}\cdot\frac{\pi}{2tR}\big(1-e^{-tR}\big) \longrightarrow 0 \text{ as } R\to\infty.\end{aligned}$$

It follows that

$$f(t) = 2\cos 2t + \tfrac{3}{2}\sin 2t.$$