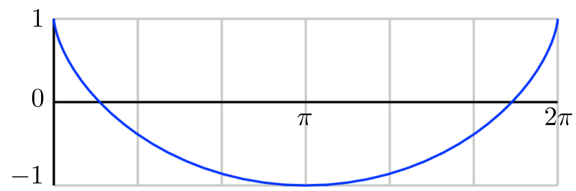
ECE 206 Advanced Calculus 2, Solutions to Assignment 11

1: (a) Let $\alpha(t) = t + i\,e^{it}$ with $0 \le t \le 2\pi$. Sketch the path α, then find its length.

Solution: We can sketch the path α by plotting points. The path is a cycloid



We have $\alpha'(t) = 1 - e^{it} = (1-\cos t) - i \sin t$, so

$$|\alpha'(t)|^2 = (1-\cos t)^2 + (\sin t)^2 = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2 - 2\cos t = 4\sin^2(t/2)\,,$$

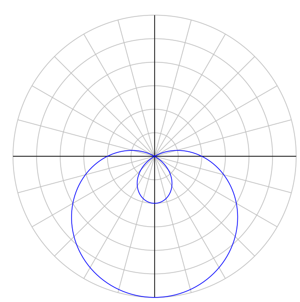
and so

$$L(\alpha) = \int_0^{2\pi} |\alpha'(t)|\,dt = 2\int_0^{\pi} 2\sin(t/2)\,dt = \Big[-8\cos(t/2)\Big]_0^{\pi} = 8\,.$$

(b) Let $\alpha(t) = (2 - 4\sin t)e^{it}$ for $0 \le t \le 2\pi$. Sketch $\alpha(t)$ and use the sketch to find the winding numbers $\eta(\alpha, a_i)$ for $a_1 = i$, $a_2 = -i$ and $a_3 = -3i$.

Solution: We make a table of values and plot points on a polar grid.

$\theta = t$	$r = 2 - 4\sin t$
0	2
$\pi/6$	0
$\pi/4$	$2 - 2\sqrt{2}$
$\pi/3$	$2 - 2\sqrt{3}$
$\pi/2$	-2



From the sketch we see that $\eta(\alpha, i) = 0$, $\eta(\alpha(-i) = 2$ and $\eta(\alpha - 3i) = 1$.

2: (a) Find $\displaystyle\int_0^{2\pi} \sin t\, e^{it}\,dt$.

Solution: We have

$$\begin{aligned}\int_0^{2\pi} (\sin t)e^{i\,t}\,dt &= \int_0^{2\pi} \sin t\,(\cos t + i\,\sin t)\,dt = \int_0^{2\pi} \sin t\cos t + i\sin^2 t\;dt \\ &= \int_0^{2\pi} \sin t\cos t\;dt + i\int_0^{2\pi} \tfrac{1}{2} - \tfrac{1}{2}\cos 2t\;dt \\ &= \Big[\tfrac{1}{2}\sin^2 t\Big]_0^{2\pi} + i\Big[\tfrac{1}{2} - \tfrac{1}{4}\sin 2t\Big]_0^{2\pi} = i\,\pi\,.\end{aligned}$$

(b) Find $\displaystyle\int_\alpha \frac{e^z\,dz}{(e^z+i)^2}$ where $\alpha(t) = i\,\frac{\pi}{4}\left(1 + e^{i\,t}\right)$ for $0 \le t \le \pi$.

Solution: Make the substitution $u = e^z + i$, $du = e^z\,dz$. Then

$$\int \frac{e^z\,dz}{(e^z+i)^2} = \int \frac{du}{u^2} = -\frac{1}{u} + c = -\frac{1}{e^z+i} + c\,.$$

By the Fundamental Theorem of Calculus,

$$\int_\alpha \frac{e^z\,dz}{(e^z+i)^2} = \left[-\frac{1}{e^z+i}\right]_{\alpha(0)}^{\alpha(\pi)} = \left[-\frac{1}{e^z+i}\right]_{i\,\pi/2}^{0} = -\tfrac{1}{1+i} + \tfrac{1}{2i} = -\tfrac{1-i}{2} - \tfrac{i}{2} = -\tfrac{1}{2}\,.$$

3: (a) Find $\displaystyle\int_\alpha z^2 - 2i\,\overline{z}\;dz$ where $\alpha(t) = -i + (1+i)t$ for $0 \le t \le 3$.

Solution: We have

$$\begin{aligned}\int_\alpha z^2 - 2i\,\overline{z}\;dz &= \int_0^3 \Big(\alpha(t)^2 - 2i\,\overline{\alpha(t)}\Big)\alpha'(t)\;dt \\ &= \int_0^3 \Big(\big(-i + (1+i)\,t\big)^2 - 2i\big(i + (1-i)\,t\big)\Big)(1+i)\,dt \\ &= (1+i)\int_0^3 -1 - 2i(1+i)\,t + 2i\,t^2 - 2i(1-i)\,t\,dt \\ &= (1+i)\int_0^3 1 - 4i\,t + 2i\,t^2\;dt \\ &= (1+i)\Big[t - 2i\,t^2 + \tfrac{2}{3}i\,t^3\Big]_0^3 \\ &= (1+i)(3 - 18i + 18i) = 3(1+i)\,.\end{aligned}$$

(b) Find $\displaystyle\int_\alpha z\sqrt{z+1}\;dz$, using the branch of the square root given by $\sqrt{r\,e^{i\theta}} = \sqrt{r}\,e^{i\,\theta/2}$ for $r > 0$ and $-\pi < \theta < \pi$, where $\alpha(t) = (1-i) + (2+i)t$ for $-1 \le t \le 1$.

Solution: To find $\int z\sqrt{z+1}\,dz$ make the substitution $w = \sqrt{z+1}$ so that $w^2 = z+1$ and $2w\,dw = dz$. Then we have

$$\begin{aligned}\int z\sqrt{z+1}\;dz = \int (w^2-1)\cdot w\cdot 2w\,dw = \int 2w^4 - 2w^2\,dw &= \tfrac{2}{5}w^5 - \tfrac{2}{3}w^3 \\ &= \big(\tfrac{2}{5}w^4 - \tfrac{2}{3}w^2\big)w = \big(\tfrac{2}{5}(z+1)^2 - \tfrac{2}{3}(z+1)\big)\sqrt{z+1}\,,\end{aligned}$$

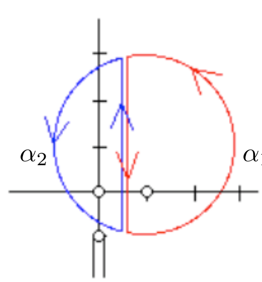
where we are using the same branch of the square root. Since $\alpha(-1) = -1-2i$ and $\alpha(1) = 3$, we have

$$\begin{aligned}\int_\alpha f &= \Big[\big(\tfrac{2}{5}(z+1)^2 - \tfrac{2}{3}(z+1)\big)\sqrt{z+1}\Big]_{-1-2i}^{3} \\ &= \big(\tfrac{2}{5}(4)^2 - \tfrac{2}{3}(4)\big)\sqrt{4} - \big(\tfrac{2}{5}(-2i)^2 - \tfrac{2}{3}(-2i)\big)\sqrt{-2i} \\ &= \big(\tfrac{32}{5} - \tfrac{8}{3}\big)(2) - \big(-\tfrac{8}{5} + \tfrac{4}{3}i\big)(1-i) = \tfrac{64}{5} - \tfrac{16}{3} + \tfrac{8}{5} - \tfrac{8}{5}i - \tfrac{4}{3}i - \tfrac{4}{3} \\ &= \big(\tfrac{72}{5} - \tfrac{20}{3}\big) - i\big(\tfrac{8}{5} + \tfrac{4}{3}\big) = \tfrac{116}{15} - \tfrac{44}{15}\,i\,.\end{aligned}$$

4: Find $\displaystyle\int_\alpha \frac{\log(z+i)}{z^2(z-1)}\,dz$ where $\log(z+i) = \ln|z+i| + i\,\theta(z+i)$ for $-\frac{\pi}{2} < \theta(z+i) < \frac{3\pi}{2}$ and $\alpha(t) = (1+i) + 2e^{it}$ with $0 \le t \le 2\pi$.

Solution: Let $f(z) = \dfrac{\log(z+i)}{z^2(z-1)}$, and let α_1 and α_2 be loops as shown below. Also, let $F(z) = \dfrac{\log(z+i)}{z^2}$ and $G(z) = \dfrac{\log(z+i)}{z-1}$ so $G'(z) = \dfrac{(z-1)/(z+i) - \log(z+i)}{(z-1)^2}$. Then, by Cauchy's Integral Formulas, we have

$$\begin{aligned}\int_\alpha f = \int_{\alpha_1} f + \int_{\alpha_2} f &= \int_{\alpha_1} \frac{F(z)}{z-1}\,dz + \int_{\alpha_2} \frac{G(z)}{z^2}\,dz = 2\pi i\big(F(1) + G'(0)\big) \\ &= 2\pi i\big(\log(1+i) + i - \log(i)\big) = 2\pi i\big(\ln\sqrt{2} + i\tfrac{\pi}{4} + i - i\tfrac{\pi}{2}\big) \\ &= \tfrac{\pi^2}{2} - 2\pi + i\,\pi\ln 2\,.\end{aligned}$$



5: Find $\displaystyle\int_\alpha \frac{dz}{z(z^2+1)^2}$ where $\alpha(t) = i + \sqrt{2}\,e^{it}$ with $\frac{5\pi}{4} \le t \le \frac{7\pi}{4}$.

Solution: Note that $z(z^2+1)^2 = z(z-i)^2(z+i)^2$. From a sketch of the curve α (it is an arc of the circle of radius $\sqrt{2}$ centred at i which starts at the point -1 and winds counterclockwise around the origin ending at the point 1) we see that $\eta(\alpha,0) = \frac{1}{2}$, $\eta(\alpha,-i) = -\frac{1}{4}$ and $\eta(\alpha,i) = \frac{1}{4}$. Now we make a partial fractions decomposition. In order to get

we need
$$\frac{1}{z(z^2+1)^2} = \frac{A}{z} + \frac{B}{z+i} + \frac{C}{(z+i)^2} + \frac{D}{z-i} + \frac{E}{(z-i)^2}$$
$$1 = A(z+i)^2(z-i)^2 + Bz(z+i)(z-i)^2 + Cz(z-i)^2 + Dz(z+i)^2(z-i) + Ez(z+i)^2\,.$$

Put in $z=0$ to get $1 = A(i)^2(-i)^2 = A$, so $A=1$. Put in $z=i$ to get $1 = E(i)(2i)^2 = -4i\,E$ so $E = \frac{1}{4}i$. Put in $z=-i$ to get $1 = C(-i)(-2i)^2 = 4i\,C$ so $C = -\frac{1}{4}i$. Equate the coefficients of z to get $0 = -iB - C + iD - E = -iB + iD$ so we see that $B = D$. Then equate the coefficients of z^4 to get $0 = A+B+D$ so $B+D=-1$ and so we have $B = D = -\frac{1}{2}$. Thus

$$\int_\alpha \frac{dz}{z(z^2+1)^2} = \int_\alpha \frac{1}{z} - \frac{\frac{1}{2}}{z+i} - \frac{\frac{1}{4}\,i}{(z+i)^2} - \frac{\frac{1}{2}}{z-i} + \frac{\frac{1}{4}\,i}{(z-i)^2}\,dz\,.$$

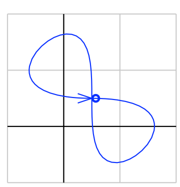
By the Winding Number Theorem, we have

$$\int_\alpha \frac{dz}{z} = \Big[\ln|z|\Big]_{-1}^{1} + 2\pi\,i\,\eta(\alpha,0) = \ln 1 - \ln 1 + \pi\,i = \pi\,i\,,$$
$$\int_\alpha \frac{dz}{z+i} = \Big[\ln|z+i|\Big]_{-1}^{1} + 2\pi\,i\,\eta(-i) = \ln\sqrt{2} - \ln\sqrt{2} - \tfrac{\pi}{2}\,i = -\tfrac{\pi}{2}\,i\ \text{, and}$$
$$\int_\alpha \frac{dz}{z-i} = \Big[\ln|z-i|\Big] + 2\pi\,i\,\eta(\alpha,i) = \ln\sqrt{2} - \ln\sqrt{2} + \tfrac{\pi}{2}\,i = \tfrac{\pi}{2}\,i\,,$$

so we have

$$\begin{aligned}\int_\alpha \frac{dz}{z(z^2+1)^2} &= \int_\alpha \frac{dz}{z} - \frac{1}{2}\int_\alpha \frac{dz}{z+i} - \frac{i}{4}\int_\alpha \frac{dz}{(z+i)^2} - \frac{1}{2}\int_\alpha \frac{dz}{z-i} + \frac{i}{4}\int_\alpha \frac{dz}{(z-i)^2}\\ &= (\pi\,i) - \tfrac{1}{2}\big(-\tfrac{\pi}{2}\,i\big) - \tfrac{1}{4}\,i\Big[-\frac{1}{z+i}\Big]_{-1}^{1} - \tfrac{1}{2}\big(\tfrac{\pi}{2}\,i\big) + \tfrac{1}{4}i\Big[-\frac{1}{z-i}\Big]_{-1}^{1}\\ &= \pi\,i - \tfrac{1}{4}\,i\left(-\tfrac{1}{1+i} + \tfrac{1}{-1+i}\right) + \tfrac{1}{4}\,i\left(-\tfrac{1}{1-i} + \tfrac{1}{-1-i}\right) = \pi\,i\,.\end{aligned}$$

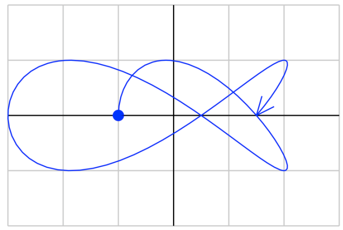
6: Find $\displaystyle\int_\alpha \frac{\sin(\pi z)}{z^4-1}\,dz$, where $\alpha(t) = \frac{1}{2}(1+i) + 6(t^2-1)t^3 + i\,4(t^2-1)^2 t$ for $-1 \le t \le 1$. The loop α is shown below.



Solution: Let $f(z) = \dfrac{\sin(\pi z)}{z^4+1}$. Note that f is undefined when $z^4+1=0$, that is when $z = \pm 1, \pm i$. Let $\alpha_1(t) = \alpha(t)$ with $-1 \le t \le 0$ and $\alpha_2(t) = \alpha(t)$ with $0 \le t \le 1$, so that α_1 loops clockwise around $z=1$ and α_2 loops counterclockwise around $z=i$. Then setting $F(z) = \dfrac{\sin(\pi z)}{(z^2+1)(z+1)}$ and $G(z) = \dfrac{\sin(\pi z)}{(z+i)(z^2-1)}$ we have

$$\begin{aligned}\int_\alpha f &= \int_{\alpha_1} f + \int_{\alpha_2} f = \int_{\alpha_1} \frac{F(z)}{z-1}\,dz + \int_{\alpha_2} \frac{G(z)}{z-i}\,dz\\ &= 2\pi i\big(-F(1) + G(i)\big) = 2\pi i\left(0 + \frac{\sin(i\pi)}{(2i)(-2)}\right) = -i\,\tfrac{\pi}{2}\sinh\pi\,.\end{aligned}$$

7: Find $\displaystyle\int_\alpha \frac{dz}{z^2(z-1)(z^2-2z+2)}$ where $\alpha(t)=\sin t+2\cos(2t)+i\,\cos(3t)$ with $-\frac{3\pi}{2}\le t\le\frac{\pi}{6}$. The path α is shown below. It starts at the point -1 and ends at the point $\frac{3}{2}$.



Solution: To get

$$\frac{1}{z^2(z-1)(z^2-2z+2)}=\frac{A}{z}+\frac{B}{z^2}+\frac{C}{z-1}+\frac{Dz+E}{z^2-2z+2}$$

we need

$$A(z)(z-1)(z^2-2z+2)+B(z-1)(z^2-2z+2)+C(z^2)(z^2-2z+2)+(Dz+E)(z^2)(z-1)=1\,.$$

Putting in $x=1$ gives $C=1$, and equating coefficients gives $A+C+D=0$, $-3A+B-2C-D+E=0$, $4A-3B+2C-E=0$, $-2A+4B=0$, and $-2B=1$. Solve these to get $A=-1$, $B=-\frac{1}{2}$, $C=1$, $D=0$ and $E=-\frac{1}{2}$, so we have

$$\frac{1}{z^2(z-1)(z^2-2z+2)}=-\frac{1}{z}-\frac{\frac{1}{2}}{z^2}+\frac{1}{z-1}-\frac{\frac{1}{2}}{z^2-2z+2}\,.$$

We can further decompose the last term. To get $\dfrac{-\frac{1}{2}}{z^2-2z+2}=\dfrac{F}{z-(1+i)}+\dfrac{G}{z-(1-i)}$ we must have $F(z-(1-i))+G(z-(1-i))=-\frac{1}{2}$. Put in $z=1+i$ to get $2i\,F=-\frac{1}{2}$ so $F=\frac{1}{4}\,i$, and put in $z=1-i$ to get $-2i\,G=-\frac{1}{2}$ so $G=-\frac{1}{4}\,i$. Thus

$$\frac{1}{z^2(z-1)(z^2-2z+2)}=-\frac{1}{z}-\frac{\frac{1}{2}}{z^2}+\frac{1}{z-1}+\frac{\frac{1}{4}\,i}{z-(1+i)}-\frac{\frac{1}{4}\,i}{z-(1-i)}\,.$$

From the picture of the path α, we see that $\eta(\alpha,0)=\frac{1}{2}$, $\eta(\alpha,1)=-\frac{3}{2}$, $\eta(\alpha,1+i)=\frac{1}{4}$ and $\eta(\alpha,1-i)=-\frac{1}{4}$, so by the Winding Number Theorem and the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_\alpha \frac{dz}{z^2(z-1)(z^2-2z+2)}&=-\int_\alpha\frac{dz}{z}-\tfrac{1}{2}\int_\alpha\frac{dz}{z^2}+\int_\alpha\frac{dz}{z-1}+\tfrac{1}{4}\,i\int_\alpha\frac{1}{z-(1+i)}-\tfrac{1}{4}\,i\int_\alpha\frac{dz}{z-(1-i)}\\
&=-\Big(\ln\tfrac{3/2}{1}+2\pi\,i\left(\tfrac{1}{2}\right)\Big)-\tfrac{1}{2}\Big[-\tfrac{1}{z}\Big]_{-1}^{3/2}+\Big(\ln\tfrac{1/2}{2}+2\pi\,i\left(-\tfrac{3}{2}\right)\Big)\\
&\quad+\tfrac{1}{4}\,i\Big(\ln\tfrac{\sqrt{5}/2}{\sqrt{5}}+2\pi\,i\left(\tfrac{1}{4}\right)\Big)-\tfrac{1}{4}\,i\Big(\ln\tfrac{\sqrt{5}/2}{\sqrt{5}}+2\pi\,i\left(-\tfrac{1}{4}\right)\Big)\\
&=-\ln\tfrac{3}{2}-\pi\,i+\tfrac{5}{6}+\ln\tfrac{1}{4}-3\pi\,i+\tfrac{1}{4}\,i\,\ln 2-\tfrac{\pi}{8}-\tfrac{1}{4}\,i\ln\tfrac{1}{2}-\tfrac{\pi}{8}\\
&=\left(\tfrac{5}{6}-\tfrac{\pi}{4}-\ln 6\right)-i\,4\pi\,.
\end{aligned}$$