ECE 206 Advanced Calculus 2, Solutions to Assignment 10

1: (a) Let $f(z)=e^{z^2}/z$. Find $f'(i)$ and $f'\left(\frac{\sqrt{\pi}}{2}(1+i)\right)$.

Solution: We have $f'(z)=\dfrac{2z^2e^{z^2}-e^{z^2}}{z^2}=\left(2-\frac{1}{z^2}\right)e^{z^2}$ and so we have $f'(i)=\left(2-\frac{1}{-1}\right)e^{-1}=3\,e^{-1}$ and $f'\left(\frac{\sqrt{\pi}}{2}(1+i)\right)=\left(2-\frac{1}{i\,\pi/2}\right)e^{i\,\pi/2}=\left(2+\frac{2}{\pi}\,i\right)\,i=-\frac{2}{\pi}+2i$.

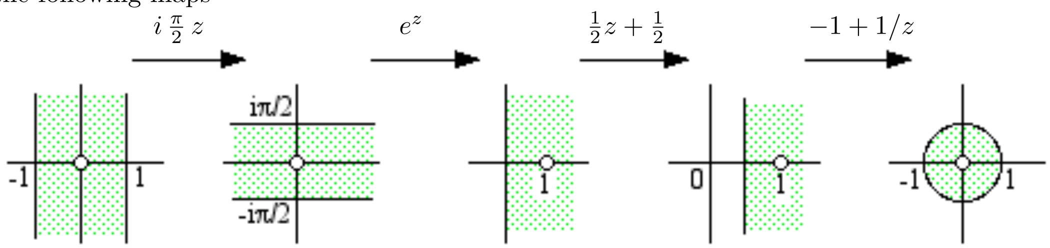
(b) Let $f\left(r\,e^{i\,\theta}\right)=\sqrt{r}\,e^{i\,\theta/2}$ where $r>0$ and $0<\theta<2\pi$. Find $f'(2i)$ and $f'(3-4i)$.

Solution: Note that f is one branch of the square root function. Differentiating both sides of $f(z)^2=z$ gives $2f(z)f'(z)=1$, and so we have $f'(z)=\dfrac{1}{2f(z)}$. Note that $f(2i)=1+i$ and $f(3-4i)=-2+i$ and so we have $f'(2i)=\frac{1}{2\sqrt{2i}}=\frac{1}{2(1+i)}=\frac{1-i}{4}$ and $f'(3-4i)=\frac{1}{2(-2+i)}=-\frac{2+i}{10}$.

2: A long thin straight wire lies along the z-axis between two grounded plates which lie along the planes $x=\pm 1$. The wire carries a uniformly distributed charge of charge density ρ.

(a) Find the electric potential u at all points between the two plates.

Solution: By symmetry, the potential does not depend on the value of z, so it suffices to find the potential at all points in the xy-plane in the set $U=\left\{x+iy\,\middle|\,-1<x<1,(x,y)\neq(0,0)\right\}$. From now on, instead of using z to denote the coordinate in the vertical direction, we shall write $z=x+iy$. Let f be the composite of the following maps



We take $u(z)=v(f(z))$ where $v(w)=-2kq\ln|w|$ and where

$$w=f(z)=-1+\frac{1}{\frac{1}{2}e^{i\,\frac{\pi}{2}z}+\frac{1}{2}}=-1+\frac{2}{e^{i\,\frac{\pi}{2}z}+1}=\frac{1-e^{i\,\frac{\pi}{2}z}}{1+e^{i\,\frac{\pi}{2}z}}=\frac{e^{-i\frac{\pi}{4}z}-e^{i\frac{\pi}{4}z}}{e^{-i\frac{\pi}{4}z}+e^{i\frac{\pi}{4}z}}=-i\tan\left(\tfrac{\pi}{4}z\right),$$

so we have $u(z)=-2kq\,\ln\left|\tan\left(\frac{\pi}{4}z\right)\right|$.

(b) Find the electric field E at all points between the plates.

Solution: Notice that $u(z)=\mathrm{Re}(g(z))$ where $g(z)=-2kq\log\left(\tan(\frac{\pi}{4}z)\right)$, using any branch of the logarithm. So we have $E(z)=-\nabla u(z)=-\overline{g'(z)}=2kq\,\overline{\dfrac{\sec^2(\frac{\pi}{4}z)}{\tan(\frac{\pi}{4}z)}\dfrac{\pi}{4}}=\overline{\dfrac{\pi kq}{\sin(\frac{\pi}{2}z)}}=\dfrac{\pi kq}{\sin(\frac{\pi}{2}\overline{z})}$.

(c) Find the direction of the electric field E at the point $(x,y,z)=\left(\frac{1}{3},\frac{\ln 4}{\pi},0\right)$.

Solution: Still writing $z=x+iy$ instead of using z to denote the vertical direction, and denoting the electric field E at the point $\left(\frac{1}{3},\frac{\ln 4}{\pi},0\right)$ by $E(z)$ with $z=x+iy=\frac{1}{3}+\frac{\ln 4}{\pi}\,i$ so that $\frac{\pi}{2}\overline{z}=\frac{\pi}{6}-i\ln 2$, we have

$$E(z)=\frac{\pi kq}{\sin(\frac{\pi}{2}\overline{z})}=\frac{\pi kq}{\sin\left(\frac{\pi}{6}-i\ln 2\right)}=\frac{kq}{\frac{1}{2}\,\frac{5}{4}-i\,\frac{\sqrt{3}}{2}\,\frac{3}{4}}=\frac{8kq}{5-3\sqrt{3}\,i}=\frac{2kq}{13}(5+3\sqrt{3}\,i)$$

and so E is in the direction of $5+3\sqrt{3}\,i$.

3: Let f be the inverse of the restriction of $\sin z$ to the set $\left\{x+i\,y\,\middle|\,-\frac{\pi}{2}<x<\frac{\pi}{2}\right\}$. Find $f'\left(\frac{4}{3}\,i\right)$ and $f'\left(-\frac{5+3i}{4\sqrt{2}}\right)$.

Solution: By differentiating both sides of the equlity $\sin\big(f(z)\big)=z$ we see that $\cos\big(f(z)\big)f'(z)=1$ so that $f'(z)=\dfrac{1}{\cos\big(f(z)\big)}$ for all z. Let $\sin^{-1}z$ denote the multi-valued inverse sine function. Recall that $\sin^{-1}z=-i\log\left(iz+\sqrt{1-z^2}\right)$, so we have

$$\begin{aligned}
\sin^{-1}\left(\tfrac{4}{3}\,i\right)&=-i\log\left(-\tfrac{4}{3}\pm\sqrt{1+\tfrac{16}{9}}\right)=-i\log\left(-\tfrac{4}{3}\pm\tfrac{5}{3}\right)=-i\log\left(\tfrac{1}{3}\right),-i\log(-3)\\
&=-i\left(\ln\tfrac{1}{3}+i\,2\pi\,k\right),-i\left(\ln 3+i(\pi+2\pi\,k)\right)=2\pi\,k+i\,\ln 3,\ \pi+2\pi\,k-i\,\ln 3\ \text{and}\\
\sin^{-1}\left(-\tfrac{5+3i}{4\sqrt{2}}\right)&=-i\log\left(\tfrac{3-5i}{4\sqrt{2}}+\sqrt{1-\tfrac{16+30i}{32}}\right)=-i\log\left(\tfrac{3-5i}{4\sqrt{2}}+\tfrac{\sqrt{16-30i}}{4\sqrt{2}}\right)=-i\log\left(\tfrac{3-5i}{4\sqrt{2}}\pm\tfrac{5-3i}{4\sqrt{2}}\right)\\
&=-i\log\left(\tfrac{8-8i}{4\sqrt{2}}\right),-i\log\left(\tfrac{-2-2i}{4\sqrt{2}}\right)=-i\log\left(2\,e^{-i\pi/4}\right),-i\log\left(\tfrac{1}{2}\,e^{i\,5\pi/4}\right)\\
&=-i\left(\ln 2+i\,(-\pi/4+2\pi\,k)\right),-i\left(\ln\tfrac{1}{2}+i\left(\tfrac{5\pi}{4}+2\pi\,k\right)\right)\\
&=-\tfrac{\pi}{4}+2\pi\,k-i\ln 2\,,\ \tfrac{5\pi}{4}+2\pi\,k+i\ln 2\,.
\end{aligned}$$

Since $f\left(\frac{4}{3}\,i\right)$ and $f\left(-\frac{5+3i}{4\sqrt{2}}\right)$ must lie in the set $\left\{x+iy\,\middle|\,-\frac{\pi}{2}<x<\frac{\pi}{2}\right\}$, it follows that $f\left(\frac{4}{3}\,i\right)=i\ln 3$ and $f\left(-\frac{5+3i}{4\sqrt{2}}\right)=-+\frac{\pi}{4}-i\ln 2$. Thus

$$\begin{aligned}
f'\left(\tfrac{4}{3}\,i\right)&=\frac{1}{\cos(i\ln 3)}=\frac{1}{\cosh(\ln 3)}=\frac{1}{\frac{3+\frac{1}{3}}{2}}=\tfrac{3}{5}\ \text{and}\\
f'\left(-\tfrac{5+3i}{4\sqrt{2}}\right)&=\frac{1}{\cos\left(-\frac{\pi}{4}-i\ln 2\right)}=\frac{1}{\cos\left(\frac{\pi}{4}+i\ln 2\right)}=\frac{1}{\frac{1}{\sqrt{2}}\cdot\frac{5}{4}-i\,\frac{1}{\sqrt{2}}\cdot\frac{3}{4}}=\tfrac{4\sqrt{2}}{5-3i}=\tfrac{2\sqrt{2}}{17}(5+3i)\,.
\end{aligned}$$

4: (a) Find the temperature $u(z)$ at each point $z\in\mathbf{C}$ given that the temperature along the positive real axis is held constant with $u(x)=40$ for $0\le x<1$ and $u(x)=10$ for $x>1$.

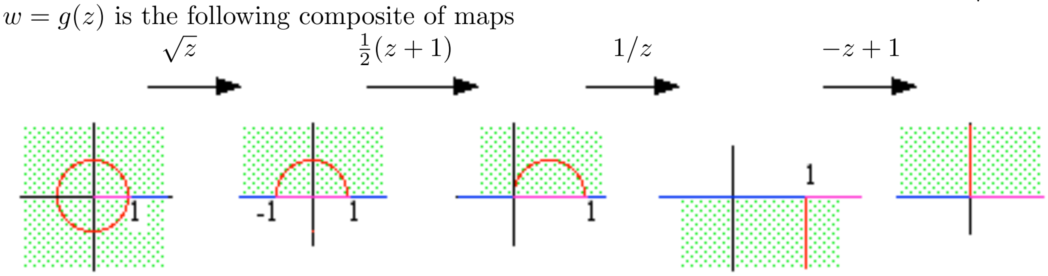
Solution: The map $f(z)=\sqrt{z}$, where $\sqrt{r\,e^{i\,\theta}}=\sqrt{r}\,e^{i\theta/2}$ with $r>0$ and $0<\theta<2\pi$, maps $\mathbf{C}\setminus\{x\in\mathbf{R}|x\ge 0\}$ to the upper half plane. The positive real axis is mapped to the entire real axis; the interval $0\le x<1$ is mapped to the interval $|x|<1$ and the interval $x>1$ is mapped to the set $|x|>1$. We wish to find a harmonic map $v(w)$, defined for w in the upper half plane, with $w(x)=10$ when $x>1$, $w(x)=40$ when $-1<x<1$ and $v(x)=10$ when $x<-1$. We can take $v(w)=v_1(w)+v_2(w)$ where $v_1(w)=10+\frac{30}{\pi}\theta(w-1)$ (so that $v_1(x)=10$ when $x>1$ and $v_1(x)=40$ when $x<1$) and $v_2(w)=-\frac{30}{\pi}\theta(w+1)$ (so that $v_2(x)=0$ when $x>-1$ and $v_2(x)=-30$ when $x<-1$). Thus the temperature is given by

$$u(z)=v\big(f(z)\big)=v_1(\sqrt{z})+v_2(\sqrt{z})=10+\frac{30}{\pi}\theta(\sqrt{z}-1)-\frac{30}{\pi}\theta(\sqrt{z}+1)=10+\frac{30}{\pi}\theta\left(\frac{\sqrt{z}-1}{\sqrt{z}+1}\right)$$

with $0\le\theta(\sqrt{z}+1)\le\theta(\sqrt{z}-1)\le\pi$.

(b) Find the isotherm $u=25$.

Solution: $u(z)=25\Longrightarrow 10+\frac{30}{\pi}\theta\left(\frac{\sqrt{z}-1}{\sqrt{z}+1}\right)=25\Longrightarrow\theta\left(\frac{\sqrt{z}-1}{\sqrt{z}+1}\right)=\frac{\pi}{2}$. Write $w=g(z)=\dfrac{\sqrt{z}-1}{\sqrt{z}+1}=1-\dfrac{2}{\sqrt{z}+1}$ so $w=g(z)$ is the following composite of maps



The isotherm $u=25$ is the image of the ray $\theta(w)=\frac{\pi}{2}$ under the inverse map $z=g^{-1}(w)$; it is the circle $|z|=1$, as seen in the above diagram.

5: One metal plate lies along the surface $\{(x,y,z)\big|x^2+y^2=1, y>0\}$ and another lies along the surface $\{(x,y,z)\big|x^2+(y+1)^2=2, y>0\}$. The first plate is held at a constant potential of 1 and the second is held at a constant potential of 4 (they are separated by insulating material along the lines $x=\pm 1$, $y=0$).

(a) Find the electric potential u at all points between the two plates.

Solution: As in Problem 2, by symmetry, the potential does not depend on the value of the vertical position z, so from now on, instead of using z to denote the vertical position, we shall write $z=x+iy$ and find the potential at all points in the set $U=\{z\in\mathbf{C}\big|\,|z|<1, |z+i|>\sqrt{2}\}$. Let $z_1=f_1(z)=z+1$ and $w=f_2(z_1)=\dfrac{1}{z_1}=\dfrac{1}{z+1}$. Then the map $f(z)=f_2\big(f_1(z)\big)=\frac{1}{z+1}$ sends the given set U to the wedge $V=\{x+i\,y\big|x>\frac{1}{2}, x+y<\frac{1}{2}\}=\{\frac{1}{2}+r\,e^{i\,\theta}\big|r>0, -\frac{\pi}{2}<\theta<-\frac{\pi}{4}\}$ and it sends the points with $|z|=1$ to the points $\frac{1}{2}+r\,e^{i\,\theta}$ with $\theta=-\frac{\pi}{2}$ and the points with $|z+i|=\sqrt{2}$ to the points $\frac{1}{2}+r\,e^{i\,\theta}$ with $\theta=-\frac{\pi}{4}$. A solution $v(w)$ to the corresponding problem in V is given by

$$v(w)=70+\tfrac{120}{\pi}\theta\big(w-\tfrac{1}{2}\big)$$

so the temperature in U is given by

$$u(z)=v(f_2(f_1(z)))=70+\tfrac{120}{\pi}\theta\Big(\tfrac{1}{z+1}-\tfrac{1}{2}\Big)=70+\tfrac{120}{\pi}\,\theta\Big(\tfrac{1-z}{2(z+1)}\Big)=70+\tfrac{120}{\pi}\theta\Big(\tfrac{1-z}{1+z}\Big)\,.$$

(b) Find the point $(0,y,0)$ where $u=3$.

Solution: For $iy\in U$ we have

$$u(i\,y)=30\iff\theta\Big(\tfrac{1-i\,y}{1+i\,y}\Big)=\tfrac{(30-70)\pi}{120}=-\tfrac{\pi}{3}\iff\theta\Big(\tfrac{(1-y^2)-i(2y)}{1+y^2}\Big)=-\tfrac{\pi}{3}\,.$$

Taking the tangent of both sides gives $\dfrac{-2y}{1-y^2}=-\sqrt{3}$, that is $\sqrt{3}y^2+2y-\sqrt{3}=0$. Using the Quadratic Formula gives $y=\frac{-2\pm\sqrt{4+12}}{2\sqrt{3}}=\frac{-1\pm 2}{\sqrt{3}}$. Since $iy\in U$, we must have $y=\frac{1}{\sqrt{3}}$.