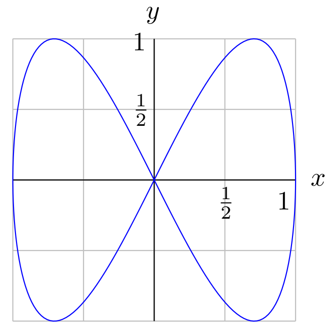
ECE 206 Advanced Calculus 2, Solutions to Assignment 1

1: Let $\mathcal{C}$ be the parametric curve $(x, y) = f(t) = (\cos t, \sin 2t)$.

(a) Make an accurate sketch of the curve $\mathcal{C}$.

Solution: We make a table of values and plot points.

t	x	y
0	1	0
$\pi/6$	$\sqrt{3}/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	1
$\pi/3$	$1/2$	$\sqrt{3}/2$
$\pi/2$	0	0
$2\pi/3$	$-1/2$	$-\sqrt{3}/2$
$3\pi/4$	$-\sqrt{2}/2$	-1
$5\pi/6$	$-\sqrt{3}/2$	$-\sqrt{3}/2$
π	-1	0
etc		



(b) Find an implicit equation for the tangent line to $\mathcal{C}$ at the point where $t = \frac{\pi}{3}$.

Solution: We have $f(t) = (\cos t, \sin 2t)$ and $f'(t) = (-\sin t, 2\cos 2t)$. The required tangent line is the line through the point $f\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in the direction of the vector $f'\left(\frac{\pi}{3}\right) = \left(-\frac{\sqrt{3}}{2}, -1\right)$. A normal vector is given by $\left(1, -\frac{\sqrt{3}}{2}\right)$, so the equation can be written as $x - \frac{\sqrt{3}}{2}y = c$. Put in the point $(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ to get $c = \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = -\frac{1}{4}$. Thus the line has equation $x - \frac{\sqrt{3}}{2}y = -\frac{1}{4}$, or equivalently $4x - 2\sqrt{3}y + 1 = 0$.

(c) Find an implicit equation for $\mathcal{C}$.

Solution: When $x = \cos t$ and $y = \sin 2t = 2\sin t \cos t$ we have

$$\begin{aligned} x^2 &= \cos^2 t \text{ , and} \\ y^2 &= 4\sin^2 t \cos^2 t = 4(1 - \cos^2 t)\cos^2 t = 4(1 - x^2)(x^2) = 4x^2 - 4x^4. \end{aligned}$$

Thus the curve is given implicitly by the equation $y^2 = 4x^2 - 4x^4$.

2: Let $\mathcal{S}$ be the parametric surface $(x,y,z) = f(s,t) = \left(\frac{s}{t}, \sqrt{s+t}, st\right)$.

(a) Find the derivative matrix $Df(s,t)$.

Solution: The derivative matrix is

$$Df(s,t) = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & -\frac{s}{t^2} \\ \frac{1}{2\sqrt{s+t}} & \frac{1}{2\sqrt{s+t}} \\ t & s \end{pmatrix}.$$

(b) Find a parametric equation for the tangent plane to $\mathcal{S}$ at the point where $(s,t) = (2,2)$.

Solution: The tangent plane is given parametrically by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = L(s,t) = f(2,2) + Df(2,2)\begin{pmatrix} s-2 \\ t-2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} \\ 2 & 2 \end{pmatrix}\begin{pmatrix} s-2 \\ t-2 \end{pmatrix}$$

that is by

$$(x,y,z) = (1,2,4) + \left(\tfrac{1}{2}, \tfrac{1}{4}, 2\right)(s-2) + \left(-\tfrac{1}{2}, \tfrac{1}{4}, 2\right)(t-2).$$

Alternatively, by introducing new parameters u and v with $s-2 = 4u$ and $t-2 = 4v$, we have

$$(x,y,z) = (1,2,4) + (2,1,8)\,u + (-2,1,8)\,v.$$

(c) Find an implicit equation for the tangent plane to $\mathcal{S}$ at the point where $(s,t) = (2,2)$.

Solution: The plane has normal vector $(2,1,8) \times (-2,1,8) = (0,-32,4)$. We can multiply this vector by $-\frac{1}{4}$ to get the simpler normal vector $(0,8,-1)$, so the equation of the plane is of the form $0x + 8y - 1z = c$ for some constant c. Put in the point $(x,y,z) = (1,2,4)$ to get $c = 12$. Thus the tangent plane is given implicitly by $8y - z = 12$ (or explicitly $z = 8y - 12$).

3: Let $\mathcal{C}$ be the curve of intersection of the two surfaces $z = x^2 - 2y$ and $z = 2x^2 + y^2$. Find a parametric equation for the tangent line $\mathcal{L}$ to the curve $\mathcal{C}$ at the point $(-1,-1,3)$ using each of the following two methods.

(a) Find the equation of the tangent plane to each of the two surfaces at $(-1,-1,3)$, then solve the two equations to obtain a parametric equation for $\mathcal{L}$.

Solution: Note that the first surface is given explicitly by $z = f(x,y) = x^2 - 2y$. We have $\frac{\partial f}{\partial x}(x,y) = 2x$ and $\frac{\partial f}{\partial y}(x,y) = -2$. The equation of the tangent plane is

$$z = f(-1,-1) + \tfrac{\partial f}{\partial x}(-1,-1)(x+1) + \tfrac{\partial f}{\partial y}(-1,-1)(y+1) = 3 - 2(x+1) - 2(y+1) = -2x - 2y - 1.$$

The second surface is given explicitly by $z = g(x,y) = 2x^2 + y^2$. We have $\frac{\partial g}{\partial x} = 4x$ and $\frac{\partial g}{\partial y} = 2y$ so the equation of the tangent plane is

$$z = g(-1,-1) + \tfrac{\partial g}{\partial x}(-1,-1)(x+1) + \tfrac{\partial g}{\partial y}(-1,-1)(y+1) = 3 - 4(x+1) - 2(y+1) = -4x - 2y - 3.$$

The equations of the two planes can be written as $2x + 2y + z = -1$ and $4x + 2y + z = -3$. We solve these two equations using Gauss-Jordan elimination. We have

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & -1 \\ 4 & 2 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

so the solution is

$$(x,y,z) = \left(-1, \tfrac{1}{2}, 0\right) + \left(0, -\tfrac{1}{2}, 1\right) t\,.$$

(b) Find a parametric equation for $\mathcal{C}$, then use this parametric equation to find a parametric equation for the tangent line $\mathcal{L}$.

Solution: For any point (x,y,z) which lies in the intersection, we must have $z = x^2 - 2y$ and $z = 2x^2 + y^2$, and so $x^2 - 2y = 2x^2 + y^2$, that is $x^2 + y^2 + 2y = 0$. Complete the square to rewrite this as $x^2 + (y+1)^2 = 1$, and we see that (x,y) lies on the circle centered at $(0,-1)$ of radius 1. This circle is given parametrically by $(x,y) = \big(\cos t, \sin t - 1\big)$. Put $x = \cos t$ and $y = \sin t - 1$ back into the equation $z = x^2 - 2y$ to get $z = \cos^2 t - 2\sin t + 2$. Thus the curve of intersection is given parametrically by

$$(x,y,z) = \big(\cos t\,,\ \sin t - 1\,,\ \cos^2 t - 2\sin t + 2\big)\,.$$

The tangent vector at each point is given by $(x', y', z') = \big(-\sin t, \cos t, -2\sin t\cos t - 2\cos t\big)$. Notice that when $t = \pi$ we have $(x,y,z) = (-1,-1,3)$ and $(x',y',z') = (0,-1,2)$, so the tangent line at the point $(x,y,z) = (-1,-1,3)$ is given parametrically by

$$(x,y,z) = (-1,-1,3) + (0,-1,2)\,t\,.$$

4: (a) Let $\mathcal{P}$ be the tangent plane to the surface given by $z = 4x^2 - 8xy + 5y^2$ at the point where $(x, y) = (2, 1)$. Find the line of intersection of $\mathcal{P}$ with the xy-plane.

Solution: The surface is given explicitly by $z = f(x, y) = 4x^2 - 8xy + 5y^2$. We have $\frac{\partial f}{\partial x} = 8x - 8y$ and $\frac{\partial f}{\partial y} = -8x + 10y$, so the equation of the tangent plane $\mathcal{P}$ is

$$z = f(2,1) + \tfrac{\partial f}{\partial x}(2,1)(x-2) + \tfrac{\partial f}{\partial y}(2,1)(y-1) = 5 + 8(x-2) - 6(y-1) = 8x - 6y - 5.$$

To find the intersection of this plane with the xy-plane, put in $z = 0$ to get $8x - 6y = 5$.

(b) Find the equation of the tangent plane to the surface given by $e^{x+z} = \sqrt{x^2y + z}$ at the point $(1, 2, -1)$.

Solution: The surface is given implicitly by $g(x, y, z) = 0$ where $g(x, y, z) = e^{x+z} - \sqrt{x^2y + z}$. We have

$$\frac{\partial g}{\partial x} = e^{x+z} - \frac{xy}{\sqrt{x^2y+z}} \; , \; \frac{\partial g}{\partial y} = -\frac{x^2}{2\sqrt{x^2y+z}} \; \text{ and } \; \frac{\partial g}{\partial z} = e^{x+z} - \frac{1}{2\sqrt{x^2y+z}}$$

so that

$$\tfrac{\partial g}{\partial x}(1,2,-1) = e^0 - \tfrac{2}{\sqrt{1}} = -1 \; , \; \tfrac{\partial g}{\partial y}(1,2,-1) = -\tfrac{1}{2\sqrt{1}} = -\tfrac{1}{2} \; \text{ and } \; \tfrac{\partial g}{\partial z}(1,2,-1) = e^0 - \tfrac{1}{2\sqrt{1}} = \tfrac{1}{2}.$$

Thus the equation of the tangent plane is

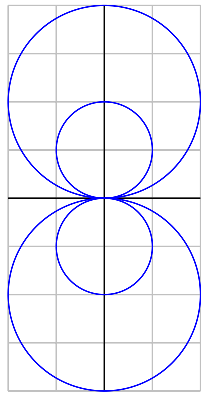
$$0 = \tfrac{\partial g}{\partial x}(1,2,-1)(x-1) + \tfrac{\partial g}{\partial y}(1,2,-1)(y-2) + \tfrac{\partial g}{\partial z}(1,2,-1)(z+1) = -(x-1) - \tfrac{1}{2}(y-2) + \tfrac{1}{2}(z+1).$$

Multiply both sides by -2 to get $0 = 2(x-1) + (y-2) + (z+1) = 2x + y - z - 5$. Thus the tangent plane is given implicitly by $2x + y - z = 5$ (or explicitly by $z = 2x + y - 5$).

5: Let $\mathcal{S}$ be the surface $2yz = x^2 + y^2$.

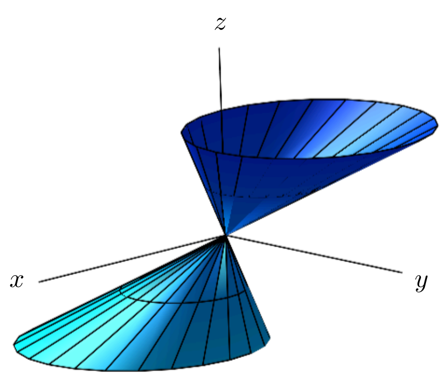
(a) Sketch the level sets $z = -2, -1, 0, 1, 2$ for the surface $\mathcal{S}$ (in other words, sketch the curve of intersection of $\mathcal{S}$ with the each of the planes $z = -2, -1, 0, 1, 2$).

Solution: The level set $z = -2$ is the curve $x^2 + y^2 = -4y$, that is $x^2 + y^2 + 4y = 0$ or, by completing the square, $x^2 + (y+2)^2 = 4$, so it is the circle centered at $(0, -2)$ of radius 2. In general, the level curve $z = c$ is the curve $x^2 + y^2 - 2cy = 0$ or $x^2 + (y-c)^2 = c^2$, which is the circle centered at $(0, c)$ of radius $|c|$. When $c = 0$, the level set consists only of the origin. The level sets are shown below.



(b) Sketch the surface $\mathcal{S}$.

Solution: To sketch the surface, we draw each of the level sets $z = c$ at height c. It also helps to find the level sets $x = 0$ and $y = 0$. When $x = 0$ (that is in the yz-plane) we get the curve $2yz = y^2$, that is $y^2 - 2yz = 0$ or $y(y - 2z) = 0$, which is the union of the two lines $y = 0$ and $y = 2z$ in the yz-plane. When $y = 0$ (that is in the xz-plane) we get $x^2 = 0$, that is the line $x = 0$ in the xz-plane.



(c) Find the equation of the tangent plane to $\mathcal{S}$ at the point $(3, 1, 5)$.

Solution: Note that $\mathcal{S}$ is given implicitly by $g(x, y, z) = 0$ where $g(x, y, z) = x^2 + y^2 - 2yz$ and that we have $g(3, 1, 5) = 0$. We have $Dg = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = (2x, 2y - 2z, -2y)$ so that $Dg(3, 1, 5) = (6, -4, -2)$. The equation of the tangent plane is

$$0 = Dg(3,1,5)\begin{pmatrix} x-3 \\ y-1 \\ z-5 \end{pmatrix} = 6(x-3) - 4(y-1) - 2(z-5) = 6x - 4y - 2z - 4.$$

We can also write the equation explicitly as $z = 3x - 2y - 2$.

6: The position of a fly at time t is given by $(x, y, z) = (t, t^2, 1 + t^3)$. A light shines down on the fly from the point $(0, 0, 3)$ and casts a shadow on the xy-plane. Find the position and the velocity of the shadow at time $t = 1$.

Solution: When the fly is at the point (x, y, z), with $z < 3$, let us find a formula for the position $(u, v, 0)$ of the shadow. The line from the light at $(0, 0, 3)$ to the fly at (x, y, z) has parametric equation

$$(u, v, w) = (0, 0, 3) + s\big((x, y, z) - (0, 0, 3)\big) = \big(sx\,,\ sy\,,\ 3 + s(z - 3)\big)\,.$$

The shadow is at the point where this line touches the xy-plane, that is the point where $w = 0$. To get $w = 0$, we need $3 + s(z - 3) = 0$, and so $s = \dfrac{3}{3 - z}$, and then $u = sx = \dfrac{3x}{3 - z}$ and $v = st = \dfrac{3y}{3 - z}$. This shows that when the fly is at the point $(x, y, z) = (t, t^2, 1 + t^3)$, the shadow is at the point

$$(u, v) = (u(t), v(t)) = \left(\frac{3x}{3 - z}, \frac{3y}{3 - z}\right) = \left(\frac{3t}{2 - t^3}, \frac{3t^2}{2 - t^3}\right)$$

and its velocity is

$$(u'(t), v'(t)) = \left(\frac{(3)(2 - t^3) - (3t)(-3t^2)}{(2 - t^3)^2}, \frac{(6t)(2 - t^3) - (3t^2)(-3t^2)}{(2 - t^3)^2}\right) = \left(\frac{6 + 6t^3}{(2 - t^3)^2}, \frac{12t + 3t^4}{(2 - t^3)^2}\right)\,.$$

In particular, we have $\big(u(1), v(1)\big) = (3, 3)$ and $\big(u'(1), v'(1)\big) = (12, 15)$.