Solutions to the Problems Using Invariants or Monovariants

1: Show that if 25 people play in a ping pong tournament then, at the end of the tournament, the number of people who played an even number of games is odd.

Solution: Each game involves two people. If n_i is the number of games played by the i^{th} player and n is the total number of games played, then we have $n_1 + n_2 + \cdots + n_{25} = 2n$. Since the sum $n_1 + n_2 + \cdots + n_{25}$ is even, an even number of the numbers n_i must be odd and so an odd number of the n_i will be even. (The invariant is the parity of $n_1 + n_2 + \cdots + n_{25}$).

2: Show that in a house with 25 rooms, if every room has an odd number of doors then there must be an odd number of doors along the outside wall of the house.

Solution: We suppose that only doors between two rooms and doors along the outside wall are being counted (so for example, closet doors do not count). If we treat the exterior of the house as a room, then each door connects two rooms. For $i = 1, 2, \cdots, 25$ let n_i be the number of doors in the i^{th} room, let n_0 be the number of doors along the outside wall, and let n be the total number of doors in the house. Then $n_0 + n_1 + \cdots + n_{25} = 2n$, which is even, and so an even number of the 26 numbers n_i must be odd. Since the 25 numbers $n_1, n_2, \cdots, n_{25}$ are all odd, the last number n_0 must also be odd.

3: Find the minimum number of breaks required to break an $m \times n$ bar of chocolate into 1×1 squares.

Solution: Each break increases the number of pieces by one. We start with one piece and we end with mn pieces and so $mn - 1$ breaks are required. (The invariant here is the number of pieces minus the number of breaks).

4: We begin with the numbers $1, 2, 3, \cdots, 50$ written on the blackboard. At each step we can erase any two of the numbers a and b and then write down the number $|a-b|$. We continue until one number remains. Determine whether this final number could be equal to 10.

Solution: The invariant we use is the parity of the sum of all the numbers on the board. Replacing two numbers a and b by the number $|a - b|$ does not change the parity of the sum of the numbers. Since $1 + 2 + \cdots + 50$ is odd, the final number remaining must be odd.

5: Let A be a matrix with integral entries. Show that there exist diagonal matrices B and C whose diagonal entries are all ± 1 such that in the matrix BAC the sum of the entries in each row and in each column is non-negative.

Solution: Multiplying on the left by matrix B is equivalent to multiplying several columns by -1, and multiplying on the right by R is equivalent to multiplying several rows by -1. Perform the following algorithm on the matrix A. If the matrix has a row or a column whose sum is negative, choose such a row or column and multiply it by -1. Repeat this operation indefinitely. Notice that at each step the sum of all the entries in the resulting matrix will increase, and this sum is always bounded by the sum of the absolute values of all the entries in the original matrix A. So after finitely many steps we will obtain a matrix with no rows or columns whose sum is negative. (Here, the sum of the entries in the matrix is a monovariant, that is a quantity which increases under an operation).

6: Determine whether it is possible to tile a 10×10 square floor using 1×4 rectangular tiles.

Solution: It is not possible. To see this, divide the floor into one hundred 1×1 squares, then assign an element of $\mathbf{Z}_4$ to each of these squares as follows: the square in the i^{th} row and j^{th} column is assigned the element $i+j \in \mathbf{Z}_4$. Notice that when a 1×4 tile is placed in any position, it will always cover 4 squares which have been assigned 4 different elements of $\mathbf{Z}_4$. To cover the entire floor, we would need 25 tiles, so each of the 4 elements of $\mathbf{Z}_4$ would need to be assigned to exactly 25 of the 100 squares. But $1 \in \mathbf{Z}_4$ is only assigned to 24 of the squares while $3 \in \mathbf{Z}_4$ is assigned to 26 of the squares. (Alternatively, this follows easily from problem 7).

7: We try to tile a $k \times l$ rectangular floor using some 2×2 square tiles and some 1×4 rectangular tiles. Show that if it is possible to tile the floor using m of the square tiles and n of the rectangular tiles, then it is not possible to tile the floor using $(m+1)$ of the square tiles and $(n-1)$ of the rectangular tiles.

Solution: Divide the floor into 1×1 squares. Label each square by an element of $\mathbf{Z}_2$ as follows: the $(i,j)^{th}$ square is labeled by 1 if i and j are both even and by 0 otherwise. Notice that when a 2×2 tile is placed in any position, it covers exactly one square labeled by 1, and when a 1×4 tile is placed in any position it covers either zero or two squares labeled by 1. Thus the sum in $\mathbf{Z}_2$ of all the labels of all the 1×1 squares covered by m square tiles and n rectangular tiles is $m + 2n$, while for $(m+1)$ square tiles and $(n-1)$ rectangular tiles, the sum is $(m+1) + 2(n-1) = m + 2n - 1$.

8: Show that when a 6×6 square floor is tiled using 1×2 rectangular tiles, there is always a straight line which crosses the floor without cutting through any of the tiles.

Solution: There are 10 lines which could possibly cross through the square without cutting any tiles (5 horizontal lines and 5 vertical lines). Notice that each line must cut through an even number of tiles, since if it cut through an odd number of tiles then, excluding those tiles, the area of the remaining portion of the floor on each side of the line would be odd and could not be tiled by 1×2 tiles. Suppose, for a contradiction, that each of the 10 lines crosses a tile. Then (since each line crosses an even number of tiles), each of the 10 lines crosses at least 2 tiles and so (since each tile is crossed by a unique line) there are at least 20 tiles. But the floor is 6×6 so there are only 18 tiles.

9: Initially, 9 of the 100 squares in a 10×10 grid are infected. During each unit time interval, each square which has 2 or more infected neighbours (a neighbour being a square which shares an edge) also becomes infected. Determine whether it is possible that all 100 squares will eventually become infected.

Solution: It is not possible. Consider the perimeter of the union of all of the infected squares. It is not hard to check (with a few pictures) that when a square with two infected neighbours becomes infected this perimeter remains constant, when a square with 3 infected neighbours becomes infected this perimeter decreases by 2, and when a square with 4 infected neighbours becomes infected this perimeter decreases by 4. Thus the perimeter never increases. The perimeter of the union of the initial 9 infected squares is at most 36, but the perimeter of the entire 10×10 grid is 40. (The perimeter is a monovariant).

10: Initially, 4 chips are placed at the point $(0,0)$. At each step we can remove one chip from some point (a,b) and replace it with 2 chips, one at the point $(a+1,b)$ and the other at $(a,b+1)$. Show that, after finitely many steps, there will always be some point with at least two chips sitting on it.

Solution: We assign a weight to each chip as follows: a chip at the point (a,b) is assigned the weight $\frac{1}{2^{a+b}}$. We consider the sum of the weights of all the chips. Notice that two chips, one at $(a+1,b)$ and the other at $(a,b+1)$ have combined weight $\frac{1}{2^{a+1+b}}+\frac{1}{2^{a+b+1}}=\frac{1}{2^{a+b}}$, so the sum of the weights of the chips is invariant. Initially, the sum of the weights is 4, so the sum will always be equal to 4 after each step.

If we had infinitely many chips with one at every point in the non-negative quadrant (that is at each point (a,b) with $a \geq 0$ and $b \geq 0$), then the sum of the weights would be $\sum\limits_{a=0}^{\infty}\sum\limits_{b=0}^{\infty}\frac{1}{2^{a+b}}=\sum\limits_{a=0}^{\infty}\frac{1}{2^a}\sum\limits_{b=1}^{\infty}\frac{1}{2^b}=2\cdot 2=4$. After finitely many steps we will have finitely many chips in the non-negative quadrant. If all these chips were at distinct points, then the sum of the weights would be less than 4. Thus there will always be some point with two chips.

Solutions to the Problems Using the Extremal Method

1: Show that every polyhedron has at least two faces with the same number of edges.

Solution: Choose a face with a maximal number of edges, say it has n edges. Then it has n neighbouring faces each of which has between 3 and n edges. By the pigeonhole principle, some two of these n neighbours must have the same number of edges.

2: There are $2n$ distinct points in the plane, no three collinear. Half of them are colored blue and the other half are colored red. Show that it is possible to pair each blue dot with a red dot in such a way that no two of the n line segments between pairs intersect.

Solution: Of all the ways of pairing the blue points with the red points, choose a way which minimizes the sum of the lengths of the segments between pairs. Suppose that two of the line segments between pairs had a point of intersection, say the segment from the blue point b_1 to the red point r_1 intersected with the segment from the blue point b_2 to the red point r_2 intersected. Then by replacing the pairs (b_1, r_1) and (b_2, r_2) by the pairs (b_1, r_2) and (b_2, r_1), we would obtain a new pairing with a smaller sum of segment lengths.

3: Show that the equation $x^2 + y^2 = 3(z^2 + w^2)$ has no non-zero solution $(x, y, z, w) \in \mathbf{Z}^4$.

Solution: Suppose that the given equation did have a non-zero solution. Of all the non-zero solutions, choose one which minimizes the value of $x^2 + y^2$, say we choose the solution (a, b, c, d). So $a^2 + b^2 = 3(c^2 + d^2)$ and $a^2 + b^2$ is as small as possible. Since $a^2 + b^2 = 0 \bmod 3$, we must have $a = b = 0 \bmod 3$, say $a = 3p$ and $b = 3q$. Then $9p^2 + 9q^2 = 3(z^2 + w^2)$ so $z^2 + w^2 = 3(a^2 + b^2) = 0 \bmod 3$, and so $z = w = 0 \bmod 3$, say $z = 3r$ and $w = 3s$. But then (p, q, r, s) is also a non-zero solution to the given equation and $p^2 + q^2 < a^2 + b^2$.

4: There are $n > 3$ lines in the plane, no two of which are parallel and no three of which meet at a point. Show that the lines cut the plane into $\frac{n^2+n+2}{2}$ regions, at least $\frac{2n-2}{3}$ of which are triangles.

Solution: Rotate the plane if necessary so that none of the n lines are horizontal. Then all of the regions which are bounded below will have a unique lowest point, and each of the $\binom{n}{2}$ points of intersection of the n lines will be the lowest point of a uniquely determined region. Thus there are $\binom{n}{2}$ regions which are bounded below. To count the regions which are not bounded below, imagine a horizontal line which lies below all of the $\binom{n}{2}$ intersection points of the given lines; this line intersects each of the n given lines and we see that there are $n + 1$ regions which are not bounded below (these regions cut our imagined line into $n + 1$ parts at the n points of intersection with the given lines). Thus the total number of regions is $\binom{n}{2} + n + 1 = \frac{n^2+n+2}{2}$.

Now, let us count the triangular regions. Let r be any one of the n lines. Note that r must have some intersection points on one (or both) sides. Choose one side of r in which there are some intersection points, and choose an intersection point p which is nearest to r. Say p is the point of intersection of lines s and t. The lines r, s and t form a triangle. None of the other lines could cut through this triangle, since if they did then there would be another intersection point nearer to r than p. Thus the line r is adjacent to a triangular region which lies on the same side of r as p. If the line r has some intersection points on both sides of it, then it will be adjacent to two triangular regions, one on each side. Finally, we note that there are at most two lines which do not have some intersection points on both sides: indeed, if there were 3 such lines, then they would form a triangle surrounding all of the intersection points, but this is not possible since any other line would necessarily intersect one of the 3 lines at some point outside the triangle. To summarize, all lines but 2 each form an edge of at least two triangular regions, the other two lines each form an edge of at least 1 triangular region, and each triangle has 3 edges, so the total number of triangular regions is at least $\frac{(n-2)\cdot 2+2\cdot 1}{3} = \frac{2n-2}{3}$.

5: Find the smallest number of chips which can be placed in the squares of an $n \times n$ grid in such a way that for each (i,j) such that there is no chip in the $(i,j)^{th}$ square, the total number of chips in the i^{th} row together with the j^{th} column is at least equal to n.

Solution: We claim that the smallest number of chips is $\left\lfloor \frac{n^2+1}{2} \right\rfloor$ Suppose that chips have been placed in the $n \times n$ grid in accordance with the given requirements. From amongst all of the rows and columns, choose one with the smallest number of chips, let us say that it happens to be a row, and let k be the number of chips on this row. In our chosen row, there are k squares with a chip and $n-k$ squares with no chip. Each of the k columns through a square with a chip has at least k chips (by our choice of the row), and each of the $n-k$ columns through a square with no chip has at least $n-k$ chips (by the requirement in the statement of the problem), and so the total number of chips is at least $k^2+(n-k)^2 = \frac{n^2}{2} + \frac{(n-2k)^2}{2} \geq \left\lfloor \frac{n^2+1}{2} \right\rfloor$. Note that we can use exactly $\left\lfloor \frac{n^2+1}{2} \right\rfloor$ chips by placing the chips in a checkerboard pattern.

6: An odd number of people stand in a room and each person has one ball. Each person tosses the ball to their nearest neighbour. Show that somebody ends up without a ball.

Solution: We shall suppose that there are at least 3 people and that each person has a unique nearest neighbour. When the number of people is equal to 3, the two people who are nearest will exchange balls, and the third person will throw his ball to one of those two and end up without a ball. Suppose, inductively, that when there are $2n-1$ people in the room, somebody will end up without a ball. Now suppose that there are $2n+1$ people in the room. Consider the pair of people who are nearest together. They will throw their balls to each other. If a third person throws their ball to one of this pair, then one of the pair will end up with two balls so someone else must end up with no ball. If no one else throws their ball to either member of this pair, then we can isolate this pair and then we are left with $2n-1$ people in the room. By the induction hypothesis, someone will end up without a ball.

7: In a ping pong tournament, each player plays every other player exactly once. Show that there is some player in the unfortunate position that every other player either beat him or beat someone who did.

Solution: Choose a player, say A, who lost the most games. We claim that A is in the above-described unfortunate position. Indeed, if A was not in that unfortunate position, then there would be a player B who neither beat A nor beat anyone who beat A, but then B would have lost more games than A.

8: Two players start with several piles of chips and they take turns. At each turn the player whose turn it is divides every pile with more than one chip into two smaller piles. The player who makes the last such division, leaving a single chip in each pile, is the winner. Describe the initial situation for which the first player will win, and describe the winning strategy.

Solution: When a pile with $2^n - 1$ chips is split into two piles, the larger of the two will have m chips with $2^{n-1} - 1 < m < 2^n - 1$, so it will not have $2^k - 1$ chips for any k. On the other hand, a pile containing m chips with $2^{n-1} - 1 < m < 2^n - 1$ can be divided into two piles in such a way that the larger of the two has $2^{n-1} - 1$ chips. Thus the first player will win provided that the largest pile does not contain $2^n - 1$ chips for any n, and the wining strategy is to split the piles so that the largest pile does have $2^n - 1$ chips for some n.

9: Let S be a finite set of 3 or more points in the plane with the property that any three points in S can be covered by a triangle of area 1. Show that the entire set S can be covered by a triangle of area 4.

Solution: Choose 3 of the points, say a, b and c, which make a triangle of maximum area. Let d, e and f be the points such that a is the midpoint of ef, b is the midpoint of df and c is the midpoint of ab. We claim that all of the given points lie in (or on) the triangle def. Suppose that some point p lies outside of def. Say it lies on the other side of the line de than the point f. But then the area of abp is greater than the area of abc.

10: A finite number of polygons lie in the plane. Each pair of polygons has a point of intersection. Show that there is a line which intersects every polygon.

Solution: Say there are n polygons, $P_1, P_2, \cdots, P_n$. For $j = 1, 2, \cdots, n$ let $I_j = [a_j, b_j]$ be the closed interval obtained by projecting P_j onto the x-axis. Note that each intersection $I_j \cap I_k$ is non-empty, since each intersection $P_j \cap P_k$ is non-empty. Let a_k be the maximum of all the left endpoints a_i, and let b_l be the minimum of all the right endpoints b_i. Note that $a_k \le b_l$ since $I_k \cap I_l \neq \emptyset$, so for all j we have $a_j \le a_k \le b_l \le b_j$, and so a_k lies in every interval I_j. Thus the line $x = a_k$ intersects every polygon.