# CO 250 Intro to Optimization, Midterm Solutions, Winter 2013

[5] **1:** Maximize and minimize $z = c_0 + c^T x$ for $x \in \mathbf{R}^5$ subject to $Ax = b$ and $x \geq 0$, where

$$c_0 = -5\ ,\ c = \big(1,2,0,3,1\big)^T\ ,\ A = \begin{pmatrix} 1 & 0 & 1 & -1 & -2 \\ 2 & 1 & 2 & -1 & -3 \\ 2 & -1 & 1 & -4 & -2 \end{pmatrix}\ ,\ b = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}.$$

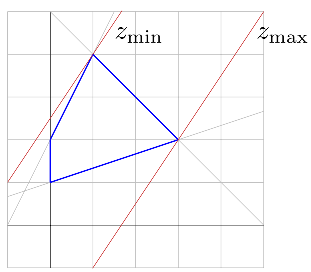
Solution: We solve $Ax = b$. We have

$$\big(A\big|b\big) = \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & -2 & -1 \\ 2 & 1 & 2 & -1 & -3 & 3 \\ 2 & -1 & 1 & -4 & -2 & -4 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & -2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 5 \\ 0 & 1 & 1 & 2 & -2 & 2 \end{array}\right)$$
$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & -2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & -3 & -3 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & -3 & -3 \end{array}\right)$$

so the solution to $Ax = b$ is given by $x = p + su + tv$ where $p = (2, 5, -3, 0, 0)^T$, $u = (2, -1, -1, 1, 0)^T$ and $v = (-1, -1, 3, 0, 1)^T$. We must optimize

$$z = c_0 + c^T x = (c_0 + c^T p) + (c^t u)s + (c^T v)t = 7 + 3s - 2t$$

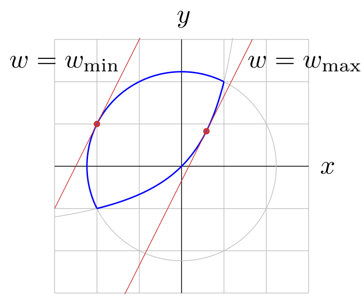
subject to the constraints $x_1 \geq 0$, $x_2 \geq 0, \cdots, x_5 \geq 0$ which we rewrite as $2s - t \geq -2$, $-s - t \geq -5$, $-s + 3t \geq 3$, $s \geq 0$ and $t \geq 0$. We draw a picture of the set of points $(s, t)$ which satisfy these constraints (outlined in blue) along with the level curves $z = \min$ and $z = \max$ (shown in orange).




We see that the maximum value is $z_{\max} = 12$ which occurs when $(s, t) = (3, 2)$, that is when $x = (6, 0, 0, 3, 2)^T$, and the minimum value is $z_{\min} = 2$ which occurs when $(s, t) = (1, 4)$, that is when $x = (0, 0, 8, 1, 4)^T$.

[5] **2:** Maximize and minimize $w = 3x - y + z$ subject to $x + z = 2$, $x^2 + y^2 \le 5$ and $2x \le y\,z$.

Solution: We solve the equality $x + z = 2$ to get $z = 2 - x$. We put this into the objective function to get $w = 3x - y + (2 - x) = 2x - y + 2$ and into the inequality constraint $2x \le y\,z$ to get $2x \le y(2 - x)$. Thus we need to maximize and minimize $w = 2 + 2x - y$ subject to $x^2 + y^2 \le 5$ and $2x \le y(2 - x)$. We draw a picture of the set of points $(x, y)$ which satisfy these inequalities, outlined in blue, along with the level curves $w = w_{\text{min}}$ and $w = w_{\text{max}}$ shown in orange. Note that $x^2 + y^2 = 5$ is the circle centred at $(0, 0)$ of radius $\sqrt{5}$ and $y = \dfrac{2x}{2 - x}$ is a hyperbola with vertical asymptote $x = 2$ and horizontal asymptote $y = \lim\limits_{x\to\infty} \dfrac{2x}{2 - x} = -2$.




We see that $w_{\text{min}}$ occurs at the point where $(x, y) = (-2, 1)$, that is at $(x, y, z) = (-2, 1, 4)$ and so we have $w_{\text{min}} = 2x - y + 2 = -3$. Also, we see that $w_{\text{max}}$ occurs at the point on the hyperbola $y = \dfrac{2x}{2 - x}$ where the slope is $m = 2$. For $y = \dfrac{2x}{2 - x}$ we have $y' = 2 \cdot \dfrac{(2 - x) + x}{(2 - x)^2} = \dfrac{4}{(2 - x)^2}$ and so $y' = 2$ when $(2 - x)^2 = 2$, that is when $x = 2 \pm \sqrt{2}$. We use the point $x = 2 - \sqrt{2}$ (the other point lies on the other branch of the hyperbola which is not shown in the above picture). Whern $x = 2 - \sqrt{2}$ we have $y = \dfrac{2x}{2 - x} = \dfrac{4 - 2\sqrt{2}}{\sqrt{2}} = 2\sqrt{2} - 2$ and $z = 2 - x = \sqrt{2}$, and so $w_{\text{max}} = 2x - y + 2 = 2(2 - \sqrt{2}) - (2\sqrt{2} - 2) + 2 = 8 - 4\sqrt{2}$.

[5] **3:** Consider the LP where we *minimize* $z = 4x_1 + 3x_2 - 2x_3$ subject to $x_1 + 2x_2 - x_3 = 1$, $3x_1 + x_2 - 2x_3 \geq 4$, $-2x_1 - x_2 + x_3 \leq -2$, $x_1 \geq 0$ and $x_2 \leq 0$.

(a) Convert the given LP into an equivalent LP in SEF for $\widetilde{x} = \big(x_1, {x_2}^-, {x_3}^+, {x_3}^-, s\,, t\,\big)^T$. Express the answer in matrix form.

Solution: We introduce variables ${x_2}^-$, ${x_3}^+$, ${x_3}^-$ with $x_2 = -{x_2}^-$ and $x_3 = {x_3}^+ - {x_3}^-$, and we introduce slack variables $s$ and $t$. We maximize $\widetilde{z} = -z = -4x_1 - 3x_2 + 2x_3 = -4x_1 + 3{x_2}^- + 2({x_3}^-{x_3}^-)$ subject to $x_1 - 2{x_2}^- - ({x_3}^+ - {x_3}^-) = 1$, $3x_1 - {x_2}^- - 2({x_3}^+ - {x_3}^-) = 4 + s$ and $-2x_1 + {x_2}^- + ({x_3}^+ - {x_3}^-) + t = -2$ with $x_1 \geq 0$, ${x_1}^- \geq 0$, ${x_3}^+ \geq 0$, ${x_3}^- \geq 0$, $s \geq 0$ and $t \geq 0$. In matrix form, we maximize $\widetilde{z} = \widetilde{c}^T \widetilde{x}$ subject to $\widetilde{A}\,\widetilde{x} = \widetilde{b}$ with $\widetilde{x} \geq 0$, where

$$\widetilde{c} = \big(-4,\, 3,\, 2, -2,\, 0,\, 0\big)^T\ ,\ \widetilde{A} = \begin{pmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 3 & -1 & -2 & 2 & -1 & 0 \\ -2 & 1 & 1 & -1 & 0 & 1 \end{pmatrix}\ ,\ \widetilde{b} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix},$$

(b) Given that $\overline{x} = (x_1, x_2, x_3)^T = (0, -1, -3)^T$ is an optimal solution to the the given LP, find an optimal solution to the equivalent LP in SEF that you obtained in part (a).

Solution: For the given solution, we have $x_1 = 0$, $x_2 = -1$ and $x_3 = -3$. For a corresponding solution $\widetilde{x} = \big(x_1, {x_2}^-, {x_3}^+, {x_3}^-,\ s,\ t\big)^T$ to the equivalent LP, we still have $x_1 = 0$, and we have ${x_2}^- = -x_2 = 1$. Also, we need ${x_3}^+ \geq 0$ and ${x_3}^- \geq 0$ with ${x_3}^+ - {x_3}^- = x_3 = -3$, so we can take ${x_3}^+ = 0$ and ${x_3}^- = 3$. Finally, the slack variables are determined by the original two inequality constraints. The inequality constraint $3x_1 + x_2 - 2x_3 \geq 4$ was replaced by the equality constraint $3x_1 + x_2 - 2x_3 = 4 + s$ so we must have $s = 3x_1 + x_2 - 2x_3 - 4 = 1$. Similarly, the equality constraint $-2x_1 - x_2 + x_3 \leq -2$ was replaced by $-2x_1 - x_2 + x_3 + t = -2$ so we must have $t = -2 + 2x_1 + x_2 - x_3 = 0$. Thus we have

$$\widetilde{x} = \big(0, 1, 0, 3, 1, 0\big)^T.$$

[5] **4:** Consider an LP in SEF with constraints $Ax = b$ and $x \geq 0$, where

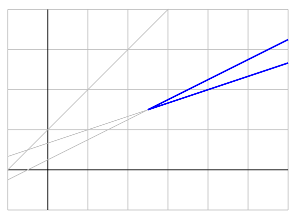
$$A = \begin{pmatrix} 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -3 \end{pmatrix} \; , \; b = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} .$$

Show that the LP is unfeasible and find a certificate of unfeasibility.

Solution: A certificate of unfeasibility is given by a vector $y$ with $b^T y < 0$ and $A^T y \geq 0$. We shall find a vector $y$ with $b^T y = -1$ and $A^T y \geq 0$. We have $b^T y = -1$ when $-y_1 + y_2 - y_3 = -1$, that is when $y_1 = 1 + y_2 - y_3$. We write the solution as $y = p + su + tv$ with $p = (1, 0, 0^T$, $u = (1, 1, 0)^T$ and $v = (-1, 0, 1)^T$. We then have

$$A^T y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -3 \end{pmatrix} \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \\ 3 \\ -4 \end{pmatrix} .$$

The constraint $A^T y \geq 0$ is equivalent to $s - t \geq -1$, $s \geq 0$, $t \geq 0$, $-s + 3t \geq 2$ and $2s - 4t \geq -1$. We draw a picture of the set of all points $(s, t)$ which satisfy these constraints, outlined in blue.



We select the point $(s, t) = (4, 2)$, which satisfies the constraints, to obtain the certificate

$$y = p + su + tv = (3, 4, 2)^T .$$

We note that because a certificate of unfeasibility exists, it follows that the given LP is unfeasible.

[5] **5:** Consider the LP where we maximize $z(x) = c_0 + c^T x$ subject to $Ax = b$ and $x \geq 0$ where

$$c_0 = 2\ ,\ c = \big(0,0,1,0,0,2\big)^T\ ,\ A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 2 & 1 & 0 & -1 \\ -3 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}\ ,\ b = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}.$$

(a) Use the Simplex Algorithm, starting with the feasible basis $\mathcal{B} = \{2, 4, 5\}$, to show that the LP is unbounded.

Solution: We perform iterations of the simplex algorithm, indicating the pivot positions in bold.

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -2 & 2 \\ -1 & 1 & \mathbf{1} & 0 & 0 & 1 & 2 \\ -1 & 0 & 2 & 1 & 0 & -1 & 4 \\ -3 & 0 & 1 & 0 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & -1 & 4 \\ -1 & 1 & 1 & 0 & 0 & 1 & 2 \\ 1 & -2 & 0 & 1 & 0 & -3 & 0 \\ -2 & -1 & 0 & 0 & 1 & \mathbf{1} & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} -3 & 0 & 0 & 0 & 1 & 0 & 5 \\ \mathbf{1} & 2 & 1 & 0 & -1 & 0 & 1 \\ -5 & -5 & 0 & 1 & 3 & 0 & 3 \\ -2 & -1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 6 & 3 & 0 & -2 & 0 & 8 \\ 1 & 2 & 1 & 0 & -1 & 0 & 1 \\ 0 & 5 & 5 & 1 & -2 & 0 & 8 \\ 0 & 3 & 2 & 0 & -1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}.$$

At this stage we see that $\widetilde{c}_5 > 0$ and $\widetilde{A}_5 \leq 0$ and so the LP is unbounded.

(b) Find a feasible point $x$ with $z(x) = 100$.

Solution: From our work in part (a) we see that a certificate of unboundedness is given by the basic point $\overline{x} = (1, 0, 0, 8, 0, 3)$ and the vector $y = (1, 0, 0, 2, 1, 1)$. We recall that for all $t \geq 0$, the point $\overline{x} + ty$ is feasible and we have

$$z(\overline{x} + ty) = c_0 + c^T(\overline{x} + y) = (c_0 + c^T\overline{x}) + t(c^T y) = z(\overline{x}) + t(c^T y) = 8 + 2t\,.$$

To get $z(\overline{x} + ty) = 100$, we choose $t = 46$, and we obtain

$$x = \overline{x} + ty = (1, 0, 0, 8, 0, 3)^T + 46\,(1, 0, 0, 2, 1, 1)^T = (47, 0, 0, 100, 46, 49)^T.$$