CO 250 Intro to Optimization, Solutions to the Midterm, Winter 2012

[5] **1:** Maximize and minimize $z = c^T x$ for $x \in \mathbf{R}^5$ subject to $Ax = b$ and $x \geq 0$, where

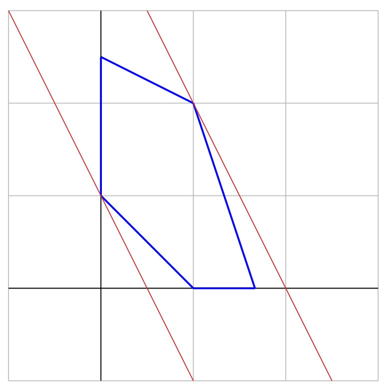
$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \ b = \begin{pmatrix} 5 \\ 5 \\ -1 \end{pmatrix}, \ c = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 3 \end{pmatrix}.$$

You may solve this using a sketch of the feasibility set.

Solution: The solution to the equation $Ax = b$ is $x = p + su + tv$ where $p = (5, 5, -1, 0, 0)$, $u = (-1, -3, 1, 1, 0)$ and $v = (-2, -1, 1, 0, 1)$. To get $x \geq 0$ we need $-s - 2t \geq -5$, $-3s - t \geq -5$, $s + t \geq 1$, $s \geq 0$ and $t \geq 0$. Also note that

$$z = c^T x = c^T (p + su + tv) = (c \cdot p) + (c \cdot u)\, s + (c \cdot v)\, t = 4 + 2s + t\,.$$

The feasible set is shown below in the st-plane, outlined in blue, along with the lines $z = \min$ and $z = \max$.



We see that the maximum value of z occurs when $(s, t) = (1, 2)$ and then we have $z_{\max} = 8$, and the minimum occurs when $(s, t) = (0, 1)$ and then we have $z_{\min} = 5$.

[5] **2:** A company produces 2 products, P_1 and P_2. Production of each product requires the use of 3 resources R_1, R_2 and R_3. Let a_{ij} be the number of units of R_i needed to produce one unit of P_j, let c_i be the cost per unit of R_i, let m_i be the maximum number of units of R_i available to the company, and let p_j be the selling price per unit of P_j. Suppose these constants are as follows.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 1 \end{pmatrix} , \; c = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} , \; m = \begin{pmatrix} 8 \\ 12 \\ 5 \end{pmatrix} , \; p = \begin{pmatrix} 10 \\ 9 \end{pmatrix} .$$

Set up an LP which could be solved to determine the number of units x_j of each product P_j the company should produce to maximize its profit (you do not need to solve the LP).

Solution: For $i = 1, 2, 3$, let r_i be the amount of resource R_i used by the company to produce x_1 units of P_1 and x_2 units of P_2. Then we have $r_i = a_{i1}x_1 + a_{i2}x_2$ for each i and so $r = Ax$. The constraints are that $r_i \le m_i$ and $x_i \ge 0$, that is $Ax \le m$ and $x \ge 0$. The company's revenue is $p_1x_1 + p_2x_2 = p^Tx$ and the cost for the resources is $c_1r_1 + c_2r_2 + c_3r_3 = c^Tr = c^TAx$, and so the company's profit is $z = p^Tx - c^TAx$. Thus we must maximize $z = (p^T - c^TA)x$ subject to the constraints $Ax \le m$ and $x \ge 0$. In particular, when A, c, m and p are as above, we have $p^T - c^TA = (10, 9) - (8, 8) = (2, 1)$ so we maximize $z = 2x_1 + x_2$ subject to the constraints $x_1 + 2x_2 \le 8$, $3x_1 + x_2 \le 12$, $x_1 + x_2 \le 5$, $x_1 \ge 0$ and $x_2 \ge 0$.

[5] **3:** Consider the LP where we maximize $z = 5 + 3x_1 + x_2 - 2x_3$ subject to $x_1 - 2x_2 + x_3 \ge 1$, $x_1 - x_2 + 3x_3 \le 4$ and $2x_1 - x_3 = 3$. Convert this LP to an LP in SEF and give the tableau for the new LP.

Solution: We write $x_i = {x_i}^+ - {x_i}^-$ for $i = 1, 2, 3$ and we introduce slack variables s, t. We must maximize $z = 5 + 3{x_1}^+ - 3{x_1}^- + {x_2}^+ - {x_2}^- - 2{x_3}^+ + 2{x_3}^-$ subject to the constraints

$$\begin{aligned} {x_1}^+ - {x_1}^- - 2{x_2}^+ + 2{x_2}^- + {x_3}^+ - {x_3}^- - s &= 1 \\ {x_1}^+ - {x_1}^- - {x_2}^+ + {x_2}^- + 3{x_3}^+ - 3{x_3}^- + t &= 4 \\ 2{x_1}^+ - 2{x_1}^- + 0{x_2}^+ - 0{x_2}^- - {x_3}^+ + {x_3}^- &= 3 . \end{aligned}$$

The tableau is

$$\begin{pmatrix} -3 & 3 & -1 & 1 & 2 & -2 & 0 & 0 & 5 \\ 1 & -1 & -2 & 2 & 1 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 3 & -3 & 0 & 1 & 4 \\ 2 & -2 & 0 & 0 & -1 & 1 & 0 & 0 & 3 \end{pmatrix} .$$

[5] **4:** Let $A = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & -3 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 2 \end{pmatrix}$.

(a) Determine whether $\overline{x} = (1,4,3,2,2)^T$ and $y = (0,1,2,1,1)^T$ form a certificate of unboundedness for the LP where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$.

Solution: For $\overline{x}$ and y to form a certificate of unboundedness, we need $x \geq 0$, $A\overline{x} = b$, $y \geq 0$ $Ay = 0$ and $c^T y > 0$. We clearly have $\overline{x} \geq 0$ and $y > 0$, and we also have

$$A\overline{x} = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = b\,,$$

$$Ay = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{, and}$$

$$c^T y = (0,0,0,-1,2) \cdot (0,1,2,1,1) = 1 > 0\,,$$

and so $\overline{x}$ and y do indeed form a certificate of unboundedness.

(b) Determine whether $\overline{x} = (0,5,0,2,1)^T$ and $y = \left(\frac{1}{2}, 0, \frac{1}{2}\right)^T$ form a certificate of optimality for the LP where we maximize $w = -z = (-c)^T x$ subject to $Ax = b$ and $x \geq 0$.

Solution: For $\overline{x}$ and y to form a certificate of optimality for this LP (which uses objective vector $-c$), we need $\overline{x} \geq 0$, $A\overline{x} = b$, $y^T b = (-c)^T$ and $y^T A \geq (-c)^T$. We clearly have $\overline{x} \geq 0$ and we have

$$A\overline{x} = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \\ 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = b\,,$$

$$y^T b = \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right) \cdot \big(1,2,-1\big) = 0 \text{ and } (-c)^T \overline{x} = (0,0,0,-1,2) \cdot (0,5,0,2,1) = 0 \text{ , and}$$

$$y^T A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & -3 \end{pmatrix} = \left(\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, -2\right) \geq \big(0,0,0,1,-2\big) = (-c)^T\,,$$

and so $\overline{x}$ and y do indeed form a certificate of optimality.

[5] **5:** Use the Simplex Algorithm, starting with the feasible basis $\mathcal{B} = \{1,4,5\}$, to find the vector $x \in \mathbf{R}^6$ which maximizes $z = c_0 + c^T x$ subject to $Ax = b$ and $x \geq 0$ where $c_0 = 1$ and

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 2 & -1 & 0 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \\ 0 \\ 2 \end{pmatrix}.$$

Solution 1: If we use the Simplex Algorithm as described in the book, then we will pivot successively at the positions $(2,2)$ then $(3,6)$ then $(2,3)$ obtaining the tableaus

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} 0 & -3 & 1 & 0 & 0 & -2 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 7 & 3 & 0 & -5 & 1 \\ 1 & 0 & -3 & -2 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 & 0 & -1 & 0 \\ 0 & 0 & -5 & -2 & 1 & 3 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & -\frac{4}{3} & -\frac{1}{3} & \frac{5}{3} & 0 & \frac{8}{3} \\ 1 & 0 & 2 & 0 & -1 & 0 & 3 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & -\frac{5}{3} & -\frac{2}{3} & \frac{1}{3} & 1 & \frac{1}{3} \end{pmatrix} \sim \begin{pmatrix} 0 & 4 & 0 & 1 & 3 & 0 & 4 \\ 1 & -6 & 0 & -2 & -3 & 0 & 1 \\ 0 & 3 & 1 & 1 & 1 & 0 & 1 \\ 0 & 5 & 0 & 1 & 2 & 1 & 2 \end{pmatrix}.$$

At this stage (since the entries in the upper-left 1×6 block are all ≥ 0), we have reached the maximum value $z = 4$ which occurs at the basic point $\overline{x} = (1,0,1,0,0,2)^T$.

Solution 2: If we use the Simplex Algorithm as described in class, then we will pivot successively at positions $(3,6)$ then $(2,3)$ obtaining

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} 0 & -3 & 1 & 0 & 0 & -2 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -1 & 0 & 2 & 0 & 3 \\ 1 & 0 & 2 & 0 & -1 & 0 & 3 \\ 0 & 3 & 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 4 & 0 & 1 & 3 & 0 & 4 \\ 1 & -6 & 0 & -2 & -3 & 0 & 1 \\ 0 & 3 & 1 & 1 & 1 & 0 & 1 \\ 0 & 5 & 0 & 1 & 2 & 1 & 2 \end{pmatrix}.$$

At this stage (since the entries in the upper-left 1×6 block are all ≥ 0), we have reached the maximum value $z = 4$ which occurs at the basic point $\overline{x} = (1,0,1,0,0,2)^T$.

[5] **6:** Consider the LP with tableau

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & -2 & 0 & -6 & 3 \\ -1 & 0 & 1 & 1 & 0 & 3 & 2 \\ 3 & 0 & 0 & 2 & 1 & 2 & 1 \\ 2 & 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

Note that, up to a permutation of the rows, the LP is in canonical form for $\mathcal{B} = \{2, 3, 5\}$.

(a) If we pivot at position $(1, 4)$, then what will the new canonical basis be?

Solution: The new basis will be $\widetilde{\mathcal{B}} = \{2, 4, 5\}$

(b) If we pivot at position $(2, 6)$, then will the new basis be feasible?

Solution: If we pivot at $(2, 6)$ then we will obtain $\widetilde{b} = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T$. Since $\widetilde{b}_3 < 0$ the new basis is not feasible.

(c) If we pivot at position $(1, 1)$, then what will the new basic point be?

Solution: If we pivot at $(1, 1)$ the we will have $\widetilde{b} = (-2, 7, 4)^T$. Since columns 1, 2 and 5 of $\widetilde{A}$ will be $\widetilde{A}_1 = e_1$, $\widetilde{A}_2 = e_3$ and $\widetilde{A}_5 = e_2$, the new basic point will be $\widetilde{x} = \big(-2, 4, 0, 0, 7, 0\big)$.

(d) If we pivot at position $(2, 1)$, then what will be the value of z at the new basic point?

Solution: If we pivot at $(2, 1)$ then the value of z at the new basic point will be $\widetilde{c}_0 = 3 - \frac{4}{3} = \frac{5}{3}$.

(e) If we pivot at position $(1, 6)$, then will the value of z at the new basic point be optimal?

Solution: If we pivot at $(1, 6)$ then although we will obtain $-\widetilde{c} = (2, 0, 2, 0, 0, 0)^T$ so that $\widetilde{c} \le 0$, we will also obtain $\widetilde{b} = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$ so we do not have $\widetilde{b} \ge 0$ and so the new basic point is not feasible, so we have no reason to expect that we have reached the maximum value of z.