Name (print): ______________________

Signature: ______________________

ID Number: ______________________

CO 250, Introduction to Optimization

Final Examination, Winter Term, 2012

University of Waterloo, UAE Campus

Instructor: Stephen New

Date: April 18, 2012

Time: 9:30 am – 12:00 noon

Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
Total	/80

Instructions:

1. Place your name, signature and ID number in the spaces provided at the top of this page.

2. This test contains 10 pages, including this cover page and a page at the end for rough work.

3. No calculators are allowed.

4. Answer all 8 questions; all questions will be given equal value.

5. Provide full explanations with all your solutions.

[10] **1:** Consider the LP where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & -2 \\ 0 & -1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \text{ and } c = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Use the Simplex Algorithm, beginning with the feasible basis $\mathcal{B} = \{1, 3, 4\}$, to determine whether the LP has an optimal solution. If so then find an optimal solution $\overline{x}$, and if not then find a certificate of unfeasibility $\overline{y}$.

[10] **2:** Consider the LP where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, where

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & 2 & 5 \end{pmatrix} \ , \ b = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \text{ and } c = (1, 2, -1, 3)^T .$$

Use Phase I of the Simplex Algorithm to determine whether the LP is feasible. If so then find a basic feasible point, and if not then find a certificate of unfeasibility.

[10] **3:** Consider the LP where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, where

$$A = \begin{pmatrix} 1 & 1 & -2 & 1 & 0 & -3 \\ 1 & 0 & 1 & 2 & 5 & -3 \\ 1 & -1 & 0 & -1 & 2 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} \text{ and } c = \big(-1, -1, 1, 0, 2, 1\big)^T.$$

Show that $\overline{x} = \big(0, 1, 0, 3, 0, 1\big)^T$ is an optimal solution and find a certificate of optimality.

[10] **4:** Consider the IP where we maximize $z = c^T x$ for $x \in \mathbf{Z}^4$ subject to $Ax = b$ and $x \geq 0$, where

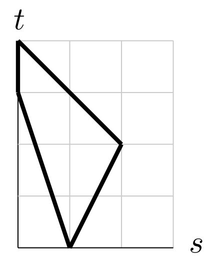
$$A = \begin{pmatrix} 1 & 1 & 4 & 2 \\ 2 & 1 & 3 & 6 \end{pmatrix} \ , \ b = \begin{pmatrix} 14 \\ 18 \end{pmatrix} \text{ and } c = \begin{pmatrix} 1 \\ 1 \\ 5 \\ 4 \end{pmatrix} .$$

Determine the duality gap by solving both the IP and its LP relaxation, using an accurate picture of the feasible set.

[10] **5:** Consider the LP where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & -2 & 0 \\ -1 & 3 & 1 & 2 & -5 \end{pmatrix} \text{ and } b = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}.$$

The solution to $Ax = b$ is $x = p + su + tv$ where $p = (4, 2, -3, 0, 0)^T$, $u = (-1, -2, 3, 1, 0)^T$ and $v = (-1, 1, 1, 0, 1)^T$. A picture of the feasible set is shown below, outlined in bold.



(a) Find every feasible basis for this LP.

(b) Find a non-zero vector c such that $\overline{x} = (0, 3, 3, 1, 3)^T$ is an optimal solution for this LP.

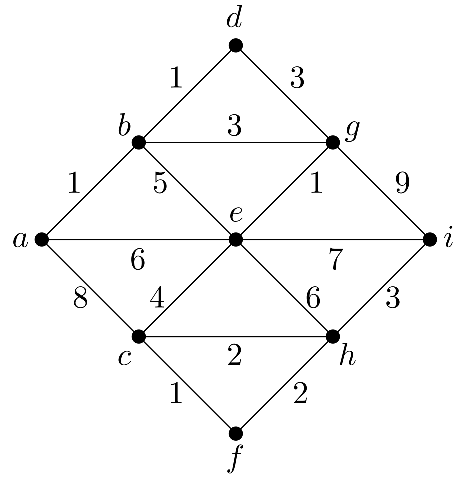
[10] **6:** Consider the LP (not in SEF) where we *minimize* $z = c^T x$ subject to $Ax \geq b$, $x \geq 0$ where

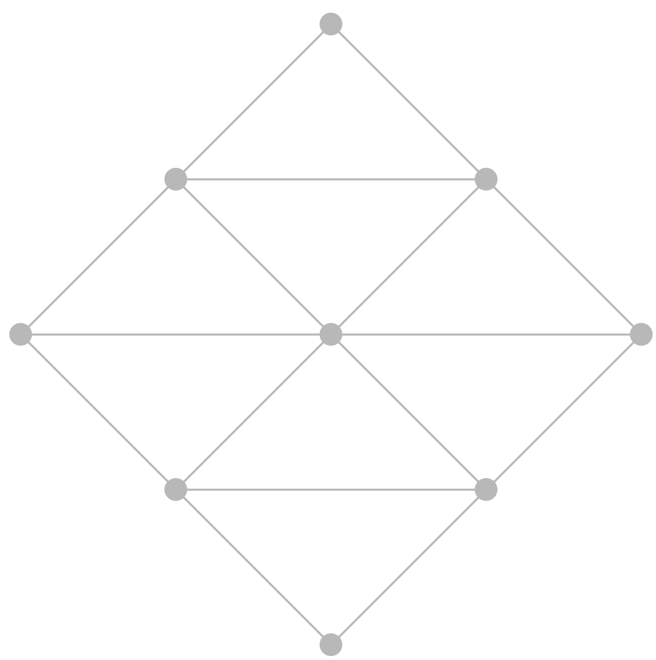
$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } c = \begin{pmatrix} 10 \\ 12 \\ 6 \end{pmatrix}.$$

(a) Show that, after replacing the dual variable y by $u = -y$, the DLP is to maximize $w = b^T u$ subject to $A^T u \leq c$ and $u \geq 0$.

(b) Given that $\overline{u} = (3, 3)^T$ is an optimal dual solution, find an optimal solution $\overline{x}$ to the original LP.

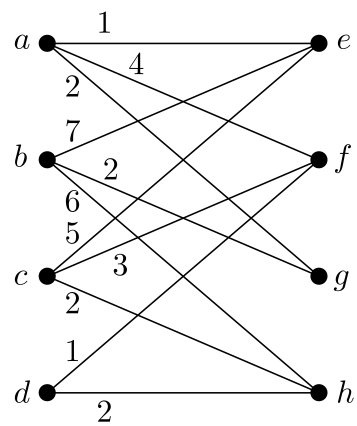
[10] **7:** (a) Use the algorithm from class to find a minimum weight path from a to i and an optimal dual solution for the following weighted graph. Show your work on the extra copy of the graph shown below.

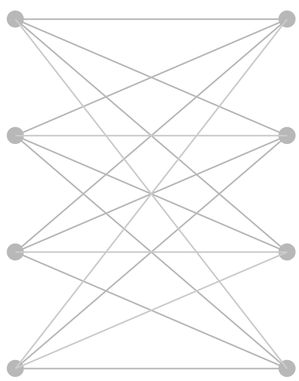


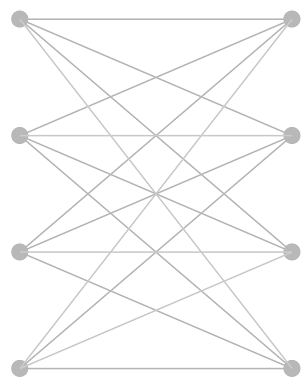


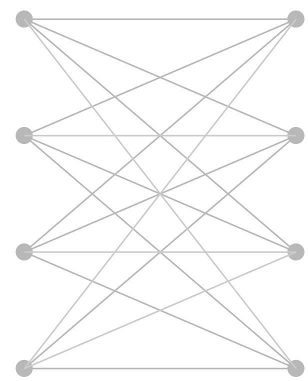
(b) For the solution x and the dual solution u that you found in part (a), state the values x_{bg}, x_{ch}, $u_{\{a,b,c\}}$ and $u_{\{a,b,d\}}$.

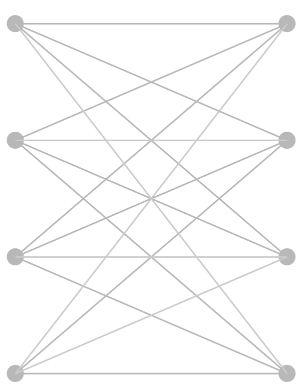
[10] **8:** Use the Hungarian Algorithm to find a maximum weight matching and an optimal dual solution for the following weighted graph. Show your work on the graphs shown below.

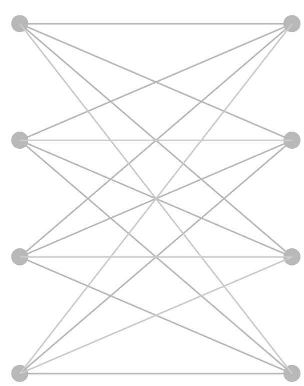


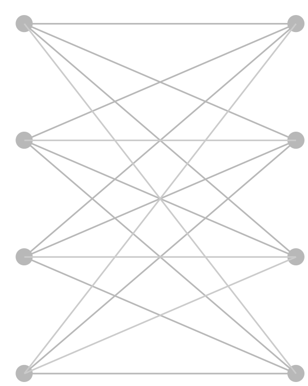












This page is for rough work. It will not be marked.