Name (print): ______________________

Signature: ______________________

ID Number: ______________________

CO 250, Introduction to Optimization

Final Examination, Winter Term, 2013

University of Waterloo, UAE Campus

Instructor: Stephen New

Date: April 11, 2013

Time: 9:30 – 12:00 am

Question	Mark
1	/5
2	/5
3	/5
4	/5
5	/5
6	/5
7	/5
Total	/35

Instructions:

1. Place your name, signature and ID number in the spaces provided at the top of this page.

2. This test contains 9 pages, including this cover page and a page at the end for rough work.

3. No calculators are allowed.

4. Answer all 7 questions; all questions will be given equal value.

5. Provide full explanations with all your solutions.

[5] **1:** Consider the IP where we maximize $z = c^T x$ for $x \in \mathbf{Z}^4$ subject to $Ax = b$ and $x \geq 0$, where

$$c = (1,0,2,2)^T \ , \ A = \begin{pmatrix} 1 & 1 & 2 & -2 \\ 2 & 1 & 1 & 1 \end{pmatrix} \ , \ b = \begin{pmatrix} 6 \\ 8 \end{pmatrix} .$$

Using an accurate picture of the feasible set, solve the IP and its LP relaxation, and hence determine the duality gap.

[5] **2:** Consider the LP where we maximize $z = c^T x$ for $x \in \mathbf{R}^4$ subject to $Ax = b$ and $x \geq 0$, where

$$c = (1, 2, -1, 1)^T \ , \ A = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 1 & -1 & 2 & 1 \end{pmatrix} \ , \ b = \begin{pmatrix} 1 \\ 3 \end{pmatrix} .$$

Use Phase I of the Simplex Algorithm to determine whether the LP is feasible. If so then find a feasible basis along with its basic point, and if not then find a certificate of unfeasibility for the original LP.

[5] **3:** Consider the LP where we maximize $z = c^T x$ for $x \in \mathbf{R}^6$ subject to $Ax = b$ and $x \geq 0$, where

$$c = \big(1, 1, 3, -1, 0, -2\big)^T \ , \ A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 1 & 0 & -2 \\ 2 & 0 & -1 & 0 & 1 & -4 \end{pmatrix} \ , \ b = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} .$$

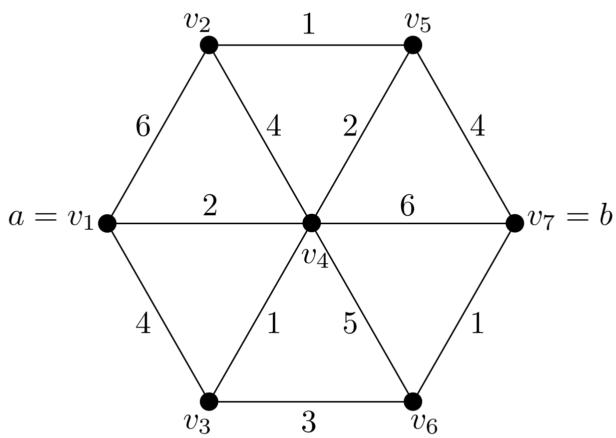
Put the LP into canonical form for the obvious feasible basis, then use Phase II of the Simplex Algorithm to determine whether the LP is unbounded or has an optimal solution. Either find a certificate of unboundedness, or find an optimal solution together with a certificate of optimality.

[5] **4:** Let $A \in M_{k\times n}(\mathbf{R})$, $B \in M_{l\times m}(\mathbf{R})$, $c \in \mathbf{R}^n$, $d \in \mathbf{R}^m$, $p \in \mathbf{R}^k$ and $q \in \mathbf{R}^l$. Consider the LP where we maximize $z = c^T x + d^T y$ for $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$, subject to the constraints $Ax \leq p$, $By = q$ and $x \geq 0$.

(a) Put the LP into SEF (using the variables $x \in \mathbf{R}^n$, $y^+ \in \mathbf{R}^m$, $y^- \in \mathbf{R}^m$ and $s \in \mathbf{R}^k$) then find and simplify the DLP, using dual variables $u \in \mathbf{R}^k$ and $v \in \mathbf{R}^l$.

(b) Let $A = I$. Show that if the LP is unbounded then d is not in the rowspace of B.

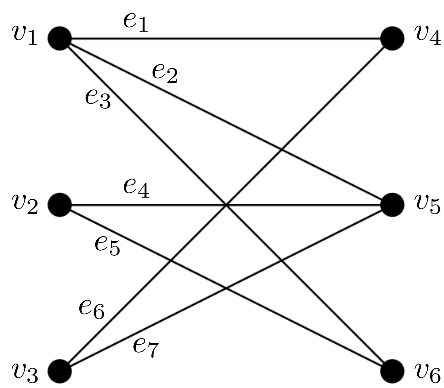
[5] **5:** Consider the weighted graph shown below.



(a) Use the Minimum Weight Path Algorithm to find a minimum weight path in G from a to b and an optimal dual solution.

(b) Let $S = \{v_1, v_4, v_5\}$ and let c be the weight vector, so $c_e = \text{weight}(e)$. Find $\sum_{e \in \text{cut}(S)} c_e$.

[5] **6:** Consider the bipartite graph shown below, with weight vector $c = \left(1, 5, 6, 2, 4, 2, 5\right)^T$.



(a) Use the Maximum Weight Perfect Matching Algorithm to find a maximum weight perfect matching, and an optimal dual solution. Express the optimal solution x (to the IP which formalizes the problem) and the optimal dual solution u in vector form.

(b) By inspection (or otherwise) find a feasible point x (for the IP which formalizes the Maximum Weight Perfect Matching Problem) which is not optimal.

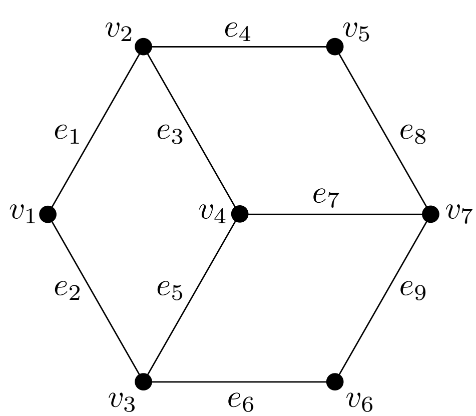
[5] **7:** Given a graph G with vertex set V and edge set E, let x be a vector with entries $x_e \in \mathbf{Z}$ for each $e \in E$. Consider the IP where we minimize $z = \sum_{e \in E} x_e$ subject to the constraints

$$\sum_{e \in E, v \in e} x_e \geq 1 \text{ for every } v \in V \text{ , and } x_e \geq 0 \text{ for every } e \in E.$$

(a) Show that (after replacing the dual variable y by $u = -y$) the DLP (the dual of the LP relaxation of the IP) is to maximize $w = \sum_{v \in V} u_v$ subject to the constraints

$$\sum_{v \in e} u_v \leq 1 \text{ for every } e \in E \text{ , and } u_v \geq 0 \text{ for every } v \in V \text{ .}$$

(b) By inspection (or otherwise) find an optimal solution x to the IP and an optimal dual solution u for the graph shown below.



This page is for rough work. It will not be marked.