CO 250 Intro to Optimization, Solutions to Assignment 6

1: Recall that we formalized the maximum weight perfect matching problem using the following LP. Given a weighted graph G, we introduce variables x_e for each edge $e \in E$, and we maximize $z = \sum_{e\in E} c_e x_e$ where $c_e = \text{weight}(e)$ subject to $\sum_{e\in E, v\in e} x_e = 1$ for each vertex v and $x_e \geq 0$ for each edge e. Using Phases I and II of the Simplex Algorithm to solve this LP, and using our formula for a certificate of optimality, find a maximum weight perfect matching and an optimal dual solution for the weighted graph G with vertex set $V = \{v_1, v_2, v_3, v_4\}$, edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ where $e_1 = \{v_1, v_2\}$, $e_2 = \{v_1, v_3\}$, $e_3 = \{v_1, v_4\}$, $e_4 = \{v_2, v_3\}$, $e_5 = \{v_2, v_4\}$, $e_6 = \{v_3, v_4\}$, and weight vector $c = \big(2, 1, 4, 3, 5, 3\big)^T$.

Solution: We need to maximize $z = c^T x$ subject to $Ax = \mathbb{1}$ and $x \geq 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Phase I of the Simplex Algorithm gives

$$\begin{pmatrix} 0 & \mathbb{1}^T & 0 \\ A & I & \mathbb{1} \end{pmatrix} = \begin{pmatrix} 0&0&0&0&0&0&1&1&1&1&0 \\ 1&1&1&0&0&0&1&0&0&0&1 \\ 1&0&0&1&1&0&0&1&0&0&1 \\ 0&1&0&1&0&1&0&0&1&0&1 \\ 0&0&1&0&1&1&0&0&0&1&1 \end{pmatrix} \sim \begin{pmatrix} -2&-2&-2&-2&-2&-2&0&0&0&0&-4 \\ \boxed{1}&1&1&0&0&0&1&0&0&0&1 \\ 1&0&0&1&1&0&0&1&0&0&1 \\ 0&1&0&1&0&1&0&0&1&0&1 \\ 0&0&1&0&1&1&0&0&0&1&1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0&0&0&-2&-2&-2&2&0&0&0&-2 \\ 1&1&1&0&0&0&1&0&0&0&1 \\ 0&-1&-1&1&1&0&-1&1&0&0&0 \\ 0&1&0&1&0&\boxed{1}&0&0&1&0&1 \\ 0&0&1&0&1&1&0&0&0&1&1 \end{pmatrix} \sim \begin{pmatrix} 0&2&0&0&-2&0&2&0&2&0&0 \\ 1&1&1&0&0&0&1&0&0&0&1 \\ 0&-1&-1&1&\boxed{1}&0&-1&1&0&0&0 \\ 0&1&0&1&0&1&0&0&1&0&1 \\ 0&-1&1&-1&1&0&0&0&-1&1&0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0&0&-2&2&0&0&0&2&2&0&0 \\ 1&1&1&0&0&0&1&0&0&0&1 \\ 0&-1&-1&1&1&0&-1&1&0&0&0 \\ 0&1&0&1&0&1&0&0&1&0&1 \\ 0&0&\boxed{2}&-2&0&0&0&0&0&1&0 \end{pmatrix} \sim \begin{pmatrix} 0&0&0&0&0&0&1&1&1&1&0 \\ 1&1&0&1&0&0&\frac12&\frac12&\frac12&-\frac12&1 \\ 0&-1&0&0&1&0&-\frac12&\frac12&-\frac12&\frac12&0 \\ 0&1&0&1&0&1&0&0&1&0&1 \\ 0&0&1&-1&0&0&\frac12&-\frac12&-\frac12&\frac12&0 \end{pmatrix}$$

We obtain the feasible basis $B = \{1, 3, 5, 6\}$. Phase II then gives

$$\begin{pmatrix} -c^T & 0 \\ A & \mathbb{1} \end{pmatrix} = \begin{pmatrix} -2&-1&-4&-3&-5&-3&0 \\ 1&1&1&0&0&0&1 \\ 1&0&0&1&1&0&1 \\ 0&1&0&1&0&1&1 \\ 0&0&1&0&1&1&1 \end{pmatrix} \sim \begin{pmatrix} -2&-1&-4&-3&-5&-3&0 \\ 1&1&0&1&0&0&1 \\ 0&-1&0&0&1&0&0 \\ 0&1&0&1&0&1&1 \\ 0&0&1&-1&0&0&0 \end{pmatrix} \sim \begin{pmatrix} 0&-1&0&-2&0&0&5 \\ 1&\boxed{1}&0&1&0&0&1 \\ 0&-1&0&0&1&0&0 \\ 0&1&0&1&0&1&1 \\ 0&0&1&-1&0&0&0 \end{pmatrix}$$

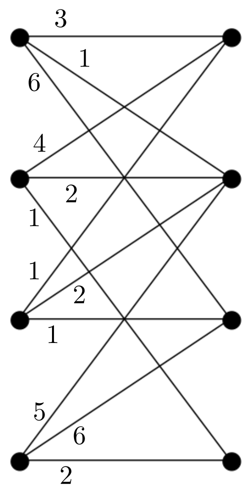
$$\sim \begin{pmatrix} 1&0&0&-1&0&0&6 \\ 1&1&0&\boxed{1}&0&0&1 \\ 1&0&0&1&1&0&1 \\ -1&0&0&0&0&1&0 \\ 0&0&1&-1&0&0&0 \end{pmatrix} \sim \begin{pmatrix} 2&1&0&0&0&0&7 \\ 1&1&0&1&0&0&1 \\ 0&-1&0&0&1&0&0 \\ -1&0&0&0&0&1&0 \\ 1&1&1&0&0&0&1 \end{pmatrix}$$

We find that an optimal solution is given by $x = (0, 0, 1, 1, 0, 0)^T$ which is the basic point for $B = \{3, 4, 5, 6\}$. The corresponding maximum weight perfect matching is $M = \{e_3, e_4\}$ with total weight $z = 4 + 3 = 7$. We have

$$\big({A_B}^T \big| c_B\big) = \left(\begin{array}{cccc|c} 1&0&0&1&4 \\ 0&1&1&0&3 \\ 0&1&0&1&5 \\ 0&0&1&1&3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1&0&0&1&4 \\ 0&1&1&0&3 \\ 0&0&1&-1&-2 \\ 0&0&1&1&3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1&0&0&1&4 \\ 0&1&0&1&5 \\ 0&0&1&-1&-2 \\ 0&0&0&2&5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1&0&0&0&\frac32 \\ 0&1&0&0&\frac52 \\ 0&0&1&0&\frac12 \\ 0&0&0&1&\frac52 \end{array}\right)$$

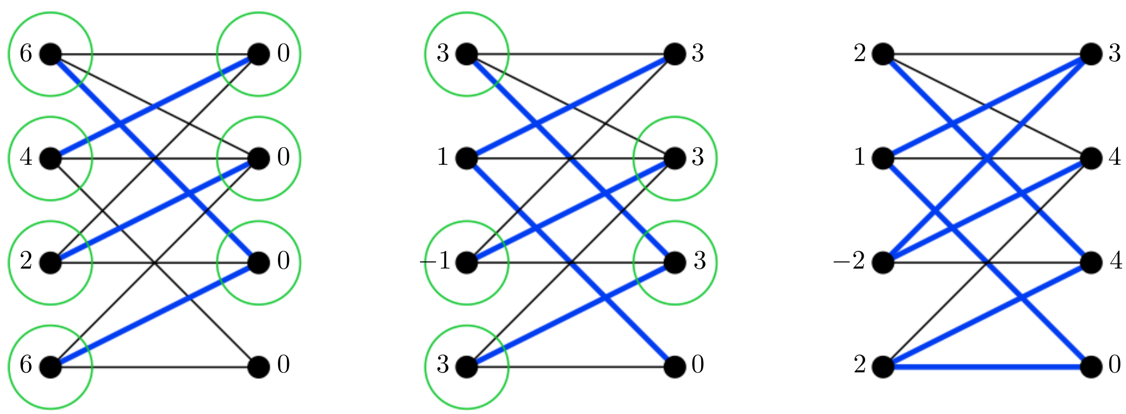
and so an optimal dual solution is given by $y = {A_B}^{-T} c_B = \big(\frac32, \frac52, \frac12, \frac52\big)^T$.

2: Consider the weighted graph shown below.



(a) Use the Maximum Weight Perfect Matching Algorithm to find a maximum weight perfect matching and and optimal dual solution.

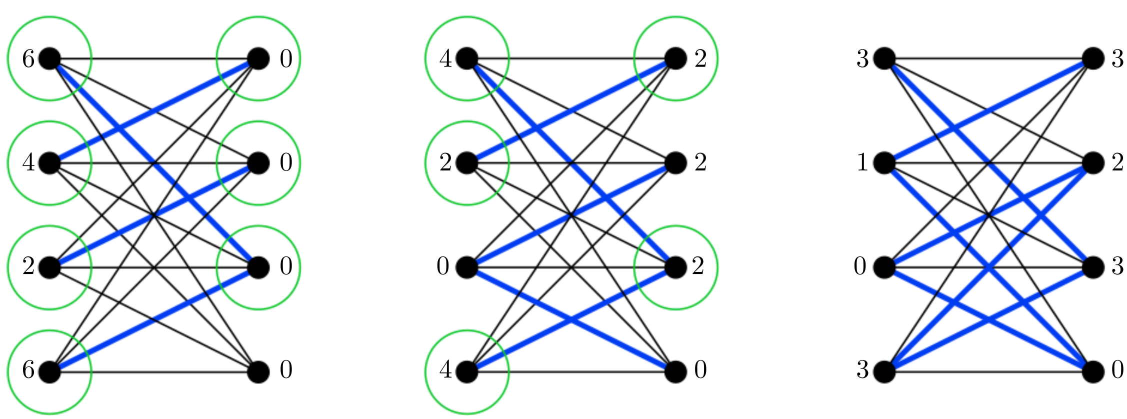
Solution: The results of the algorithm are summarized in the following sequence of pictures.



A maximum weight perfect matching can be selected from the blue edges in the final graph. For any such perfect matching, the total weight is $z = 14$.

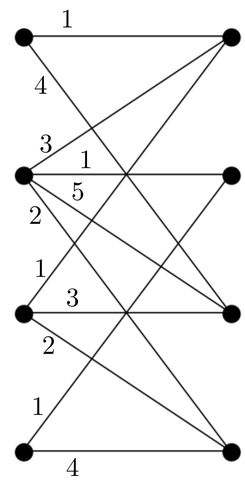
(b) Use the Maximum Weight Matching Algorithm to find a maximum weight matching and and optimal dual solution.

Solution: The results of the algorithm are summarized in the following sequence of pictures.



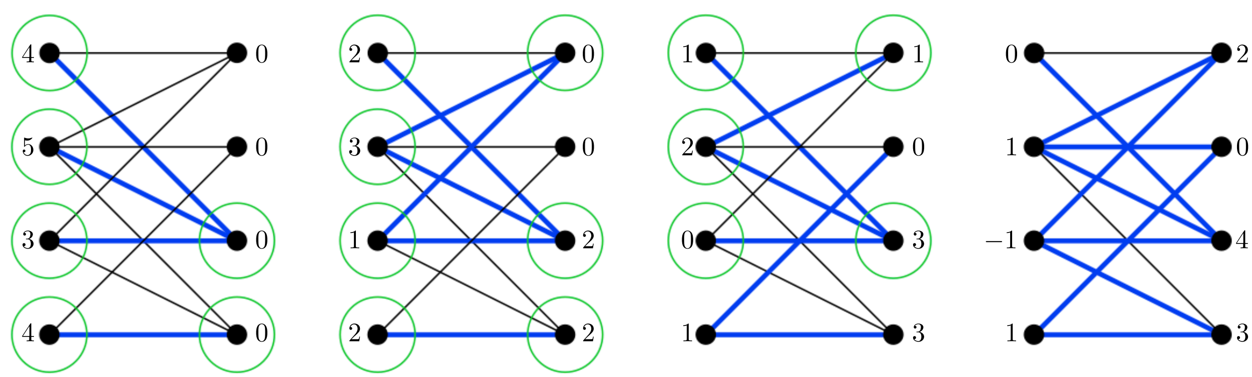
A (unique) maximum weight perfect matching can be selected from the blue edges in the final graph. The total weight is $z = 15$. Note that one of the four edges in this perfect matching was not an edge in the original graph, so the corresponding maximum weight matching for the original graph has three edges.

3: Consider the weighted graph shown below.



(a) Use the Maximum Weight Perfect Matching Algorithm to find a maximum weight perfect matching and and optimal dual solution.

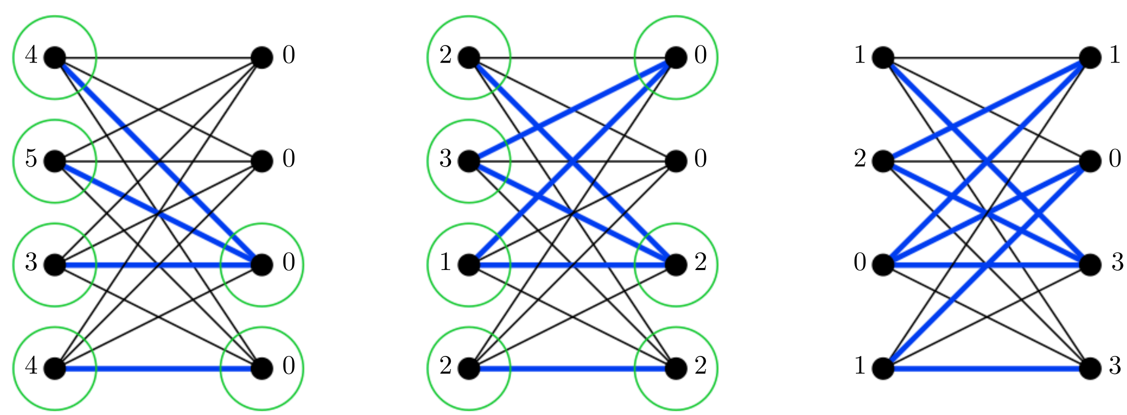
Solution: The results of the algorithm are summarized in the following sequence of pictures.



A maximum weight perfect matching can be selected from the blue edges in the final graph. For any such perfect matching, the total weight is $z = 10$.

(b) Use the Maximum Weight Matching Algorithm to find a maximum weight matching and and optimal dual solution.

Solution: The results of the algorithm are summarized in the following sequence of pictures.



A (unique) maximum weight perfect matching can be selected from the blue edges in the final graph. The total weight is $z = 11$. Note that one of the four edges in this perfect matching was not an edge in the original graph, so the corresponding maximum weight matching for the original graph has three edges.

4: Consider the IP where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, where

$$A = \begin{pmatrix} -1 & 1 & 3 & -1 \\ -2 & 1 & 4 & -3 \end{pmatrix} \ , \ b = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \ , \ c = \big(-1, 1, 4, 2 \big)^T .$$

Solve the IP, using the Simplex Algorithm repeatedly, beginning with the LP relaxation of the given IP and then finding cutting planes and adding the corresponding inequality constraints.

Solution: First we apply Phase I of the Simplex Algorithm to get

$$\begin{pmatrix} 0 & \mathbb{1}^T & 0 \\ A & I & b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 3 & -1 & 1 & 0 & 4 \\ -2 & 1 & 4 & -3 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 3 & -2 & -7 & 4 & 0 & 0 & -5 \\ -1 & 1 & 3 & -1 & 1 & 0 & 4 \\ -2 & \boxed{1} & 4 & -3 & 0 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 0 & 1 & -2 & 0 & 2 & -3 \\ \boxed{1} & 0 & -1 & 2 & 1 & -1 & 3 \\ -2 & 1 & 4 & -3 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 2 & 1 & -1 & 3 \\ 0 & 1 & 2 & 1 & 2 & -1 & 7 \end{pmatrix}$$

We obtain the feasible basis $B = \{1, 2\}$, Next we apply Phase I, first putting the tableu in canonical form for the basis $B = \{1, 2\}$.

$$\begin{pmatrix} -c^T & 0 \\ A & b \end{pmatrix} = \begin{pmatrix} 1 & -1 & -4 & -2 & 0 \\ -1 & 1 & 3 & -1 & 4 \\ -2 & 1 & 4 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & -1 & -3 & 4 \\ 1 & -1 & -3 & 1 & -4 \\ 0 & -1 & -2 & -1 & -7 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & -1 & -3 & 4 \\ 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & \boxed{2} & 1 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & \frac{1}{2} & 0 & -\frac{5}{2} & \frac{15}{2} \\ 1 & \frac{1}{2} & 0 & \boxed{\frac{5}{2}} & \frac{13}{2} \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{7}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 & 14 \\ \frac{2}{5} & \frac{1}{5} & 0 & 1 & \frac{13}{5} \\ -\frac{1}{5} & \frac{2}{5} & 1 & 0 & \frac{11}{5} \end{pmatrix}$$

The maximum is attained at a non-integer point. The first row of the final tableau gives the equality $\big(\frac{2}{5}, \frac{1}{5}, 0, 1\big)\, x = \frac{13}{5}$. We round down the coefficients and add the inequality constraint $(0, 0, 0, 1)x \leq 2$. We put the new LP (with this added constraint) into SEF by adding a slack variable s_1, and solve the new LP using Phase II, beginning with the feasible basis $\{1, 2, 5\}$.

$$\begin{pmatrix} 0 & 0 & -1 & -3 & 0 & 4 \\ 1 & 0 & -1 & 2 & 0 & 3 \\ 0 & 1 & \boxed{2} & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & \frac{1}{2} & 0 & -\frac{5}{2} & 0 & \frac{15}{2} \\ 1 & \frac{1}{2} & 0 & \frac{5}{2} & 0 & \frac{13}{2} \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{7}{2} \\ 0 & 0 & 0 & \boxed{1} & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{5}{2} & \frac{25}{2} \\ 1 & \frac{1}{2} & 0 & 0 & -\frac{5}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & 1 & 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

Again, the maximum is attained at a non-integer point. The first row of the final tableau gives the equality $\big(1, \frac{1}{2}, 0, 0, -\frac{5}{2}\big) \begin{pmatrix} x \\ s_1 \end{pmatrix} = \frac{3}{2}$. We round down the coefficients and add the inequality constraint $(1, 0, 0, 0, -3) \begin{pmatrix} x \\ s_1 \end{pmatrix} \leq 1$. We put the new LP into SEF by adding another slack variable s_2, and solve the new LP using Phase II, starting with the feasible basis $\{1, 2, 5, 6\}$.

$$\begin{pmatrix} 0 & 0 & -1 & -3 & 0 & 0 & 4 \\ 1 & 0 & -1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & -3 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & -1 & -3 & 0 & 0 & 4 \\ 1 & 0 & -1 & 2 & 0 & 0 & 3 \\ 0 & 1 & \boxed{2} & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & \frac{1}{2} & 0 & -\frac{5}{2} & 0 & 0 & \frac{15}{2} \\ 1 & \frac{1}{2} & 0 & \frac{5}{2} & 0 & 0 & \frac{13}{2} \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & \frac{7}{2} \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & -\frac{1}{2} & 0 & \boxed{\frac{1}{2}} & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 5 & 10 \\ 1 & 3 & 0 & 0 & 0 & -5 & 4 \\ 0 & 1 & 1 & 0 & 0 & -1 & 3 \\ 0 & \boxed{1} & 0 & 0 & 1 & -2 & 1 \\ 0 & -1 & 0 & 1 & 0 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 1 & 12 \\ 1 & 0 & 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 \end{pmatrix}$$

This time the maximum is attained at an integer-valued point. We see that the maximum value for the original IP is $z = 12$ and it occurs at $\begin{pmatrix} x \\ s \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \overline{s} \end{pmatrix}$ with $\overline{x} = (1, 1, 2, 2)^T$ and $\overline{s} = (0, 0)^T$.

5: Given a graph G, the *Maximum Set of Isolated Vertices Problem* is to find a set of vertices $S \subseteq V(G)$ of largest possible size such that no two vertices in S are endpoints of the same edge in G.

(a) Formulate the Maximum Set of Isolated Vertices Problem as an IP (introducing an integer variable x_v for each vertex v). Express your answer in matrix form.

Solution: Let x be the vector with integer entries x_v for each $v \in V = V(G)$. Say $n = |V|$ so that $x \in \mathbf{Z}^n$. A vector x with each $x_v \in \{0, 1\}$ corresponds to the set of vertices $S \subseteq V$ where

$$x_v = \begin{cases} 1 \text{ if } v \in S \\ 0 \text{ if } v \notin S. \end{cases}$$

We wish to maximize

$$z = |S| = \sum_{v \in S} 1 = \sum_{v \in V} x_v = \mathbb{1}^T x\,.$$

We require that each $x_v \in \mathbf{Z}$ with $x_v \geq 0$ and we require that no two vertices in S are the endpoints of the same edge, that is for every edge $e \in E = E(G)$, we require that at most one of the ends of e lies in S, or equivalently that

$$\sum_{v \in e} x_v \leq 1\,.$$

Note that this latter requirement forces each $x_v \leq 1$, so we do not need to include this constraint. In matrix form, we maximize $z = \mathbb{1}^T x$ for $x \in \mathbf{Z}^n$ subject to $Ax \leq \mathbb{1}$ and $x \geq 0$, where A is the matrix with entries

$$A_{e,v} = \begin{cases} 1 \text{ if } v \in e \\ 0 \text{ if } v \notin e\,. \end{cases}$$

(b) Find and simplify the DLP (that is the dual of the LP relaxation of the IP).

Solution: In SEF, the LPR is to maximize $z = \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}^T \begin{pmatrix} x \\ s \end{pmatrix}$ subject to $\begin{pmatrix} A & I \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = \mathbb{1}$ and $\begin{pmatrix} x \\ s \end{pmatrix} \geq 0$. The DLP is to minimize $w = \mathbb{1}^T y$ subject to $\begin{pmatrix} A^T \\ I \end{pmatrix} y \geq \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}^T$, that is $A^T \geq \mathbb{1}$ and $y \geq 0$. The dual vector y has entries $y_e \in \mathbf{R}$ for each edge $e \in E$, and the DLP is to minimize $w = \sum_{e \in E} y_e$ subject to the constraints that $y_e \geq 0$ for each edge $e \in E$ and that $\sum_{e \in E, v \in e} y_e \geq 1$ for each vertex $v \in V$.

(c) Determine the duality gap in the case of the *complete graph* on n vertices, that is the graph with n vertices and $\binom{n}{2}$ edges (so there is an edge joining every pair of vertices).

Solution: Let G be the complete graph on n vertices (with $n \geq 2$ so that the graph has an edge). Since any two vertices are joined by an edge, it follows that the largest possible set of isolated vertices consists of a single vertex. For the IP, the maximum value for z is $z_{\max} = 1$ and this occurs for any vector x which has one entry equal to 1 and all other entries equal to 0. For the LPR, we can find an optimal solution and an optimal dual solution by inspection. The vector x with entries $x_v = \frac{1}{2}$ for all $v \in V$ is feasible, and it gives $z(x) = \frac{n}{2}$ where $n = |V|$. The vector y with entries $y_e = \frac{1}{n-1}$ for every $e \in E$ is a feasible dual point since for each vector v there are exactly $n-1$ edges which have v as an endpoint so $\sum_{e \in e, v \in e} y_v = (n-1) \cdot \frac{1}{n-1} = 1$, and it gives $w(y) = \sum_{e \in E} y_e = \binom{n}{2} \frac{1}{n-1} = \frac{n}{2}$. Since $z(x) = w(y)$, we know that x and y are optimal. Thus the duality gap for this LP is equal to $\frac{n}{2} - 1$.