CO 250 Intro to Optimization, Solutions to Assignment 5

1: (a) Consider the LP where we maximize $z = 3x_1 - x_2 + 2x_3$ for $x_1, x_2, x_3 \in \mathbf{R}$ subject to the constraints $2x_1 + x_2 - x_3 \geq -4, -x_1 + 2x_2 \geq 3$, $x_1 + 3x_2 - x_3 \leq 2$, $-x_1 + 2x_2 - 2x_3 = 1$ and $x_3 \geq 0$. Put the LP into SEF using the variables ${x_1}^+, {x_1}^-, {x_2}^+, {x_2}^-, x_3, s_1, s_2, t$ and then find and simplify the DLP.

Solution: In SEF, we maximize $z = c^T x$ subject to $Ax = b$, $x \geq 0$ where $x = \left({x_1}^+, {x_1}^-, {x_2}^+, {x_2}^-, x_3, s_1, s_2, t\right)^T$ and

$$c = \left(3, -3, -1, 1, 2, 0, 0, 0\right)^T \ , \ A = \begin{pmatrix} 2 & -2 & 1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 2 & -2 & 0 & 0 & -1 & 0 \\ 1 & -1 & 3 & -3 & -1 & 0 & 0 & 1 \\ -1 & 1 & 2 & -2 & -2 & 0 & 0 & 0 \end{pmatrix} \ , \ b = \begin{pmatrix} -4 \\ 3 \\ 2 \\ 12 \end{pmatrix} .$$

The DLP is to minimize $w = b^T y$ subject to $A^T y \geq c$, that is to minimize $w = -4y_1 + 3y_2 + 2y_3 + y_4$ subject to $2y_1 - y_2 + y_3 - y_4 = 3$, $y_1 + 2y_2 + 3y_3 + 2y_4 = -1$, $-y_1 - y_3 - 2y_4 \geq 2$, $y_1 \leq 0$, $y_2 \leq 0$ and $y_3 \geq 0$.

(b) Consider the LP where we maximize $z = c^T x$ subject to $Ax \leq b$, $x \geq 0$. Put the LP into SEF then find and simplify the DLP. Show that for feasible points x and y for the LP and the DLP, the complementary slackness conditions are that for all i, either $x_i = 0$ or $(A^T y)_i = c_i$, and for all j, either $y_j = 0$ or $(Ax)_j = b_j$.

Solution: In SEF, we maximize $z = \begin{pmatrix} c \\ 0 \end{pmatrix}^T \begin{pmatrix} x \\ s \end{pmatrix}$ subject to $\begin{pmatrix} A & I \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = b$, $\begin{pmatrix} x \\ s \end{pmatrix} \geq 0$.

The DLP is to minimize $w = b^T y$ subject to $\begin{pmatrix} A^T \\ I \end{pmatrix} y \geq \begin{pmatrix} c \\ 0 \end{pmatrix}$, that is subject to $A^T y \geq c$ and $y \geq 0$.

The complementary slackness conditions are that for all i, either $\begin{pmatrix} x \\ s \end{pmatrix}_i = 0$ or $\left(\begin{pmatrix} A^T \\ I \end{pmatrix} y\right)_i = \begin{pmatrix} c \\ 0 \end{pmatrix}_i$. Equivalently, for all i either $x_i = 0$ or $\left(A^T y\right)_i = c_i$, and for all j either $s_j = 0$ or $y_j = 0$. Note that when x is feasible we have $Ax + s = b$ so that $s_j = 0$ is equivalent to $(Ax)_j = b_j$.

2: (a) Consider the LP where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, where

$$c = \left(2, 1, -3, 2, 2, 3\right)^T \ , \ A = \begin{pmatrix} 1 & 3 & 1 & 0 & -4 & 3 \\ 1 & 2 & 1 & 1 & -2 & 4 \\ 2 & 2 & -1 & 1 & -3 & 1 \end{pmatrix} \ , \ b = \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix} .$$

For each of the following points $\overline{x}$, determine whether $\overline{x}$ is an optimal solution to the LP.

$$\overline{x} = (3, 0, 1, 2, 0, 0)^T \ , \ (4, 4, 0, 0, 3, 0)^T \ , \ (1, 0, 0, 3, 0, 4)^T .$$

Solution: For $\overline{x} = (3, 0, 1, 2, 0, 0)^T$ we have $A\overline{x} = b$ and $z(\overline{x}) = c^T \overline{x} = 7$, for $\overline{x} = (4, 4, 0, 0, 3, 0)^T$ we have $A\overline{x} = b$ and $c^T \overline{x} = 18$, and for $\overline{x} = (1, 0, 0, 3, 0, 4)^T$ we have $A\overline{x} = (13, 20, 9)^T \neq b$. It follows that only the second of the three points could be an optimal solution. Note that $\overline{x} = (4, 4, 0, 0, 3, 0)^T$ is the basic point for the basis $B = \{1, 2, 5\}$. Let $y = {A_B}^{-T} c_B$. We have

$$\left({A_B}^T \middle| c_B\right) = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 3 & 2 & 2 & 1 \\ -4 & -2 & -3 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 2 & 5 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

and so $y = (-3, 5, 0)^T$. Note that $b^T y = (4, 6, 7) \cdot (-3, 5, 0) = 18 = c^T \overline{x}$ and

$$A^T y = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \\ -4 & -2 & -3 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 5 \\ 2 \\ 11 \end{pmatrix} \geq \begin{pmatrix} 2 \\ 1 \\ -3 \\ 2 \\ 2 \\ 3 \end{pmatrix} = c$$

and so $y = (-3, 5, 0)^T$ is a certificate of optimality for $\overline{x} = (4, 4, 0, 0, 3, 0)^T$.

(b) Find an example of an LP, where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, along with a basis B for the LP, such that the basic point $\overline{x}$ for B is an optimal solution to the LP, but the vector $y = {A_B}^{-T} c_B$ is not a certificate of optimality for $\overline{x}$.

Solution: We look for an example in which the optimal solution $\overline{x}$ is a basic point for several different bases. We can, for example, take $c = (0,0,1,0)^T$, $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The point $\overline{x} = (0,0,1,0)^T$ is an optimal solution and it is the basic point for each of the bases $B_1 = \{1,3\}$, $B_2 = \{2,3\}$ and $B_3 = \{3,4\}$. For $B = B_1 = \{1,3\}$, if we let

$$y = {A_B}^{-T} c_B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-T} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then we have

$$A^T y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

and so the condition that $A^T y \geq c$ is not satisfied, so y is not a certificate for $\overline{x}$.

3: Let G be the weighted graph with vertex set $V = \{v_1, v_2, v_3, v_4\}$ and edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, where $e_1 = \{v_1, v_2\}$, $e_2 = \{v_1, v_3\}$, $e_3 = \{v_1, v_4\}$, $e_4 = \{v_2, v_3\}$, $e_5 = \{v_2, v_4\}$ and $e_6 = \{v_3, v_4\}$, with weight vector $c = (2,4,5,1,3,1)^T$, where $c_i = w(e_i)$.

(a) Let $M = \{S_1, S_2, S_3, S_4\}$ where $S_1 = \{v_1\}$, $S_2 = \{v_1, v_2\}$, $S_3 = \{v_1, v_3\}$ and $S_4 = \{v_1, v_2, v_3\}$. Find $\text{cut}(S)$ for each $S \in M$, and hence find the matrix A with entries

$$A_{S,e} = \begin{cases} 1 \text{ if } e \in \text{cut}(S) \\ 0 \text{ if } e \notin \text{cut}(S) \,. \end{cases}$$

Solution: We have $\text{cut}(S_1) = \{e_1, e_2, e_3\}$, $\text{cut}(S_2) = \{e_2, e_3, e_4, e_5\}$, $\text{cut}(S_3) = \{e_1, e_3, e_4, e_6\}$, and $\text{cut}(S_4) = \{e_3, e_5, e_6\}$ and so

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

(b) To find the minimum weight path from $a = v_1$ to $b = v_4$, we minimize $z = c^T x$ subject to $Ax \geq \mathbb{1}$, $x \geq 0$. Put this LP into SEF, find and simplify the DLP replacing the dual variable y by $u = -y$, then put the DLP into SEF.

Solution: In SEF, we maximize $-z = \begin{pmatrix} -c \\ 0 \end{pmatrix}^T \begin{pmatrix} x \\ s \end{pmatrix}$ subject to $\begin{pmatrix} A & -I \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = \mathbb{1}$ with $\begin{pmatrix} x \\ s \end{pmatrix} \geq 0$.

The DLP is to minimize $-w = \mathbb{1}^T y$ subject to $\begin{pmatrix} A^T \\ -I \end{pmatrix} y \geq \begin{pmatrix} -c \\ 0 \end{pmatrix}$, that is $A^T \geq -c$ and $-y \geq 0$.

Replacing y by $u = -y$, the DLP is to maximize $w = \mathbb{1}^T u$ subject to $A^T u \leq c$ and $u \geq 0$.

In SEF, the DLP is to maximize $w = \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}^T \begin{pmatrix} u \\ t \end{pmatrix}$ subject to $\begin{pmatrix} A^T & I \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = c$ and $\begin{pmatrix} u \\ t \end{pmatrix} \geq 0$.

(c) Solve the DLP using Phase II of the Simplex Algorithm, starting with the obvious feasible basis.

Solution: The Simplex Algorithm gives

$$\begin{pmatrix} -\mathbb{1}^T & 0 & 0 \\ A^T & I & c \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 4 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & \mathbf{1} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 2 \\ 0 & 0 & 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 4 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & -1 & 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The optimal solution is the basic solution for $B = \{1, 2, 4, 6, 7, 9\}$, given by $\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} \overline{u} \\ \overline{t} \end{pmatrix}$ with $\overline{u} = (2, 1, 0, 1)$ and $\overline{t} = (0, 1, 1, 0, 1, 0)^T$.

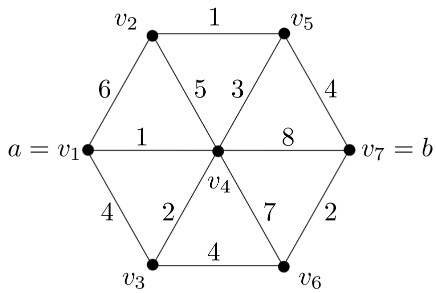
(d) Use your solution from part (c) and the formula for a certificate to obtain an optimal solution to the LP.

Solution: An optimal solution to the LP is given by a certificate $\overline{x}$ of optimality for the optimal dual solution, so we can take $\overline{x} = \left(A^T \; I\right)_B^{-T} \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}_B$. We have

$$\left((A^T \; I)_B^T \Big| \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}_B \right) = \left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

and so an optimal solution to the LP is $\overline{x} = (1, 0, 0, 1, 0, 1)^T$. This corresponds to the path with edges e_1, e_4, e_6.

4: Let G be the weighted graph with vertex set $V = \{v_1, v_2, \cdots, v_7\}$ and edge set $E = \{e_1, e_2, \cdots, e_{12}\}$, where $e_1 = \{v_1, v_2\}$, $e_2 = \{v_1, v_3\}$, $e_3 = \{v_1, v_4\}$, $e_4 = \{v_2, v_4\}$, $e_5 = \{v_2, v_5\}$, $e_6 = \{v_3, v_4\}$, $e_7 = \{v_3, v_6\}$, $e_8 = \{v_4, v_5\}$, $e_9 = \{v_4, v_6\}$, $e_{10} = \{v_4, v_7\}$, $e_{11} = \{v_5, v_7\}$ and $e_{12} = \{v_6, v_7\}$, with weight vector given by $c = (6, 4, 1, 5, 1, 2, 4, 3, 7, 8, 4, 2)^T$, where $c_i = w(e_i)$ (see the picture below).



Use the Minimum Weight Path Algorithm to find a minimum weight path from $a = v_1$ to $b = v_{12}$ along with an optimal dual solution u. At each step, indicate the vertex set S_k, the cut $\text{cut}(S_k)$, the slack $\text{sl}_k(e)$ for each $e \in \text{cut}(S_k)$, the added edge d_{k+1}, and the value of the entry u_{S_k} of the feasible dual point.

Solution: The results of the algorithm are as follows.

$S_0 = \{v_1\}$, $\text{cut}(S_0) = \{e_1, e_2, e_3\}$
$\text{sl}(e_1) = 6$, $\text{sl}(e_2) = 4$, $\text{sl}(e_3) = 1$
$d_1 = e_3 = \{v_1, v_4\}$, $u_{S_0} = \text{sl}(e_3) = 1$

$S_1 = \{v_1, v_4\}$, $\text{cut}(S_1) = \{e_1, e_2, e_4, e_6, e_8, e_9, e_10\}$
$\text{sl}(e_1) = 5$, $\text{sl}(e_2) = 3$, $\text{sl}(e_4) = 5$, $\text{sl}(e_6) = 2$, $\text{sl}(e_8) = 3$, $\text{sl}(e_9) = 7$, $\text{sl}(e_{10}) = 8$
$d_2 = e_6 = \{v_3, v_4\}$, $u_{S_1} = \text{sl}(e_6) = 2$

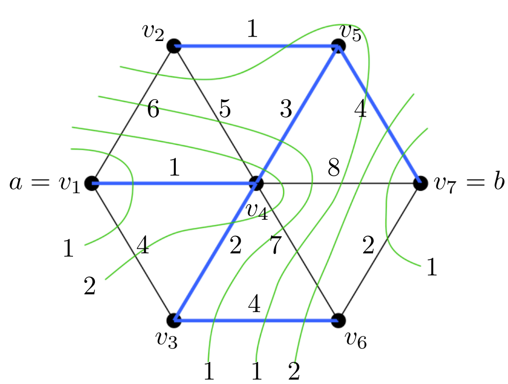
$S_2 = \{v_1, v_3, v_4\}$, $\text{cut}(S_2) = \{e_1, e_4, e_7, e_8, e_9, e_{10}\}$
$\text{sl}(e_1) = 3$, $\text{sl}(e_4) = 3$, $\text{sl}(e_7) = 4$, $\text{sl}(e_8) = 1$, $\text{sl}(e_9) = 5$, $\text{sl}(e_{10}) = 6$
$d_3 = e_8 = \{v_4, v_5\}$, $u_{S_2} = 1$

$S_3 = \{v_1, v_3, v_4, v_5\}$, $\text{cut}(S_3) = \{e_1, e_4, e_5, e_7, e_9, e_{10}, e_{11}\}$
$\text{sl}(e_1) = 4$, $\text{sl}(e_4) = 2$, $\text{sl}(e_5) = 1$, $\text{sl}(e_7) = 3$, $\text{sl}(e_9) = 4$, $\text{sl}(e_{10}) = 5$, $\text{sl}(e_{11}) = 4$
$d_4 = e_5 = \{v_2, v_5\}$, $u_{S_3} = \text{sl}(e_5) = 1$

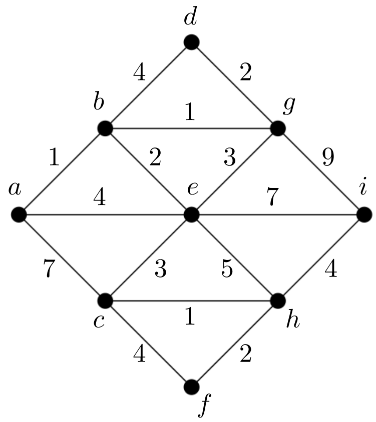
$S_4 = \{v_1, v_2, v_3, v_4, v_5\}$, $\text{cut}(S_4) = \{e_7, e_9, e_{10}, e_{11}\}$
$\text{sl}(e_7) = 2$, $\text{sl}(e_9) = 3$, $\text{sl}(e_{10}) = 4$, $\text{sl}(e_{11}) = 3$
$d_5 = e_7 = \{v_3, v_6\}$, $u_{S_4} = \text{sl}(e_7) = 2$

$S_5 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $\text{cut}(S_5) = \{e_{10}, e_{11}, e_{12}\}$
$\text{sl}(e_{10}) = 2$, $\text{sl}(e_{11}) = 1$, $\text{sl}(e_{12}) = 2$
$d_6 = e_{11} = \{v_5, v_7\}$, $u_{S_5} = \text{sl}(e_{11} = 1$.

We summarize the results in the following picture.

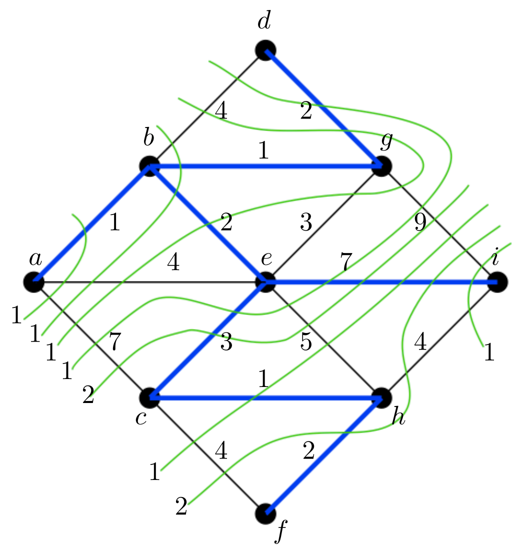


5: Let G be the weighted graph shown below.



(a) Use the Minimum Weight Path Algorithm to find a minimum weight path from a to i along with an optimal dual solution (you can indicate the steps of the algorithm in the form of a picture).

Solution: The results of the algorithm are summarized in the following picture.



We see that a minimum weight path is the path a, b, e, i with total weight $z = 1 + 2 + 7 = 10$. An optimal dual solution is as shown with value $w = 1 + 1 + 1 + 1 + 2 + 1 + 2 + 1 = 10$.

(b) Find an optimal dual solution u with as few nonzero entries u_S as you can.

Solution: An optimal dual solution u with 4 non-zero entries, namely $u_{\{a\}} = 1$, $u_{\{a,b,d,g\}} = 2$, $u_{\{a,b,d,e,g\}} = 3$ and $u_{\{a,b,c,d,e,f,g,h\}} = 4$ is shown below.