CO 250 Intro to Optimization, Solutions to Assignment 4

1: Consider an LP in SEF with constraints $Ax = b$ and $x \geq 0$, where $A \in M_{k\times n}(\mathbf{R})$, $\text{rank}\,(A) = k$ and $b \geq 0$.

(a) For $x \in \mathbf{R}^n$ with $Ax = b$, show that x is a basic point (for some basis B) if and only if $\{A_i \big| x_i \neq 0\}$ is linearly independent, where A_i denotes the i^{th} column of A.

Solution: Suppose that x is a basic point for the basis B. Since B is a basis, the set $\{A_i \big| i \in B\}$ is a basis for $\mathbf{R}^k$. Since x is the basic point for the basis B, we have $x_N = 0$ which implies that for each index i such that $x_i \neq 0$ we have $i \in B$, and so $\{A_i \big| x_i \neq 0\} \subseteq \{A_i \big| i \in B\}$. Since $\{A_i \big| x_i \neq 0\}$ is a subset of a basis for $\mathbf{R}^k$, it is linearly independent. Conversely, suppose that $\{A_i | x_i \neq 0\}$ is linearly independent. Since $\text{rank}\,(A) = k$, the set $C = \{A_i | 1 \leq i \leq n\}$ spans $\mathbf{R}^k$. Since $\{A_i | x_i \neq 0\}$ is a linearly independent subset of the spanning set C, we can find a basis S for $\mathbf{R}^k$ with $\{A_i | x_i \neq 0\} \subseteq S \subseteq C$. Since the basis S consists of k of the columns of A, we have $S = \{A_i | i \in B\}$ for some basis B of the matrix A. Since S was obtained by adding columns of A to the set $\{A_i | x_i \neq 0\}$, it follows that for every index i with $x_i \neq 0$ we have $i \in B$, and so x is the basic point for the basis B.

(b) For $x \in \mathbf{R}^n$, show that x is a basic feasible point for the LP if and only if $(x, 0)$ is a basic feasible point for the Auxiliary LP in which we maximize $w(x, s) = -\sum s_i$ subject to $Ax + s = b$ with $x \geq 0$ and $s \geq 0$.

Solution: Let $x \in \mathbf{R}^n$. Note that x is a feasible point for the LP if and only if $Ax = b$ and $x \geq 0$ if and only if $Ax + 0 = b$ and $(x, 0) \geq 0$ if and only if $(x, 0)$ is a feasible point for the auxiliary LP. Let x be a feasible point for the LP so $(x, 0)$ is a feasible point for the Auxiliary LP. Note that $\{A_i \big| x_i \neq 0\} = \{(A\,I)_i \big| (x, 0)_i \neq 0\}$, and so by part (a) we know that x is a basic point (for some basis B) if and only if $\{A_i | x_i \neq 0\}$ is linearly independent if and only if $\{(A\,I)_i \big| (x, 0)_i \neq 0\}$ is linearly independent if and only if $(x, 0)$ is a basic point for the Auxiliary LP.

(c) When $A = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 & 0 & -2 & 1 \\ 1 & 2 & 1 & 5 & 4 & 3 & 3 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$, determine which of the following points are basic points: $(1, 1, 0, 0, 0, 0, 0)^T$, $(2, -1, 2, 0, 1, 0, 0)^T$, $(1, 0, 1, 0, 1, 0, 0)^T$, $(0, 0, 1, 1, 0, 0, 0)^T$, $\left(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0, 1\right)^T$.

Solution: For $x = (1, 1, 0, 0, 0, 0, 0)^T$ we have $Ax = (2, 2, 3)^T \neq b$, so x is not a basic point (for any basis B).
For $x = (2, -1, 1, 0, 1, 0, 0)^T$ the set $\{A_i | x_i \neq 0\} = \{A_1, A_2, A_3, A_5\}$ is a set of 4 vectors in $\mathbf{R}^3$, so it must be linearly dependent, and so x is not a basic point (for any basis B).
For $x = (1, 0, 1, 0, 1, 0, 0)^T$ we do have $Ax = b$ and we have

$$\big(A_1, A_3, A_5\big) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so the set $\{A_i | x_i \neq 0\}$ is linearly independent, and so x is a basic point (indeed it is the basic point for the basis $B = \{1, 3, 5\}$).
For $x = (0, 0, 1, 1, 0, 0, 0)^T$ we do have $Ax = b$ and clearly the set $\{A_i | x_i \neq 0\} = \{A_3, A_4\}$ is linearly independent, and so x is a basic point (indeed it is the basic point for any basis of the form $B = \{A_3, A_4, A_i\}$).
For $x = \left(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0, 1\right)^T$ we have

$$\big(A_2, A_5, A_7\big) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so the set $\{A_i | x_i \neq 0\}$ is linearly dependent, and so x is not a basic point (for any basis B).

2: Consider the LP in SEF with tableau $\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix}$, where $A \in M_{k\times n}(\mathbf{R})$ and $\text{rank}\,(A) = k$.

(a) Suppose that we apply the Simplex Algorithm to the LP and obtain a basis B whose basic solution $\overline{x}$ maximizes z. Show that $\overline{x}$ together with the vector $y = {A_B}^{-T} c_B$ form a certificate of optimality for the LP.

Solution: When we apply the Simplex Algorithm, we reduce the tableau for the LP to a modified tableau $\begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}$ which is in canonical form for a feasible basis B, and the basic solution $\overline{x}$ is given by $\overline{x}_B = \widetilde{b}$ and $\overline{x}_N = 0$. In the case that the algorithm ends with an optimal solution $\overline{x}$, we have $\widetilde{c} \le 0$. Recall, from class, that $\widetilde{b} = {A_B}^{-1} b$ and $\widetilde{c} = c - A^T y$, where $y = A_B^{-T} c_B$. For y to be a certificate of optimality for $\overline{x}$ for the original LP, we need $A\overline{x} = b$, $\overline{x} \ge 0$, $c^T\overline{x} = y^T b$ and $A^T y \ge c$. We have $A\overline{x} = b$ and $\overline{x} \ge 0$ since $\overline{x}$ is a feasible point, and we have

$$c^T\overline{x} = {c_B}^T\overline{x}_B + {c_N}^T\overline{x}_N = {c_B}^T\overline{x}_B = {c_B}^T\widetilde{b} = {c_B}^T {A_B}^{-1} b = \left({A_B}^{-T} c_B\right)^T b = y^T b$$

and we have $c - A^T y = \widetilde{c} \le 0$ so that $A^T y \ge c$.

(b) Suppose that we apply the Simplex Algorithm to the LP, ending with the modified LP with tableau $\begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}$ in canonical form for the basis B, with $\widetilde{c}_l > 0$ and $\widetilde{A}_l \le 0$ (so that the original LP is unbounded). Show that the basic solution $\overline{x}$ for B together with the vector y given by $y_B = -\widetilde{A}_l$ and $y_N = (e_l)_N$ form a certificate of unboundedness for the original LP.

Solution: Recall that $\widetilde{A} = {A_B}^{-1}A$ and $\widetilde{c} = c - A^T {A_B}^{-T} c_B$, or equivalently $A = A_B\widetilde{A}$ and $c = \widetilde{c} + A^T {A_B}^{-T} c_B$. We know that $A\overline{x} = b$ and $\overline{x} \ge 0$ because $\overline{x}$ is a feasible point. Note that $y \ge 0$ because $y_B = -\widetilde{A}_l \ge 0$ and $y_N = (e_l)_N \ge 0$. We also have

$$Ay = A_B\widetilde{A}\,y = A_B\big(\widetilde{A}_B y_B + \widetilde{A}_N y_N\big) = A_B\big(I(-A_l) + \widetilde{A}_N(e_l)_N\big) = A_B(-A_l + A_l) = 0\ \text{, and}$$

$$c^T y = \left(\widetilde{c} + A^T {A_B}^{-T} c_B\right)^T y = \widetilde{c}^T y + {c_B}^T {A_B}^{-1} A y = \widetilde{c}^T y = {\widetilde{c}_B}^T y_B + {\widetilde{c}_N}^T y_N = {\widetilde{c}_N}^T y_N = \widetilde{c}_N (e_l)_N = \widetilde{c}_l > 0$$

and so $\overline{x}$ and y form a certificate of unboundedness for the original LP.

(c) Suppose that we apply Phase I of the Simplex Algorithm by solving the auxiliary LP in which we maximize $w(x,s) = -\sum s_i$ subject to $Ax + s = b$ with $x \ge 0$ and $s \ge 0$, and we obtain an optimal solution $(x,s) = (\overline{x}, \overline{s})$ with $w(\overline{x}, \overline{s}) = w_{\max} < 0$ (so the original LP is unfeasible). Show that if y is a certificate of optimality for the optimal solution $(\overline{x}, \overline{s})$ for the auxiliary LP, then the same vector y is also a certificate of unfeasibility for the original LP.

Solution: Let $u = (1, \cdots, 1)^T$ so that $w = -\sum s_i = -u^T s$ and the auxiliary LP has objective vector $(0, -u^T)$. A certificate of optimality y for the optimal solution $\begin{pmatrix} \overline{x} \\ \overline{s} \end{pmatrix}$ for this auxiliary LP satisfies the conditions $(0,-u)^T \begin{pmatrix} \overline{x} \\ \overline{s} \end{pmatrix} = y^T b$ and $y^T(A, I) \ge (0, -u^T)$. The first condition gives $y^T b = -u^T\overline{s} = -\sum \overline{s}_i = w_{\max}$ so we have $y^T b < 0$, and the second condition gives $y^T A \ge 0$ (and also $y^T \ge -u^T$). Since $y^T b < 0$ and $y^T A \ge 0$, the vector y is a certificate of unboundedness for the original LP.

3: Consider the LP in SEF with the following tableau

$$\begin{pmatrix} -1 & -2 & 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & 2 & -1 & -1 \\ 0 & 1 & 1 & 3 & -2 & -2 \\ -2 & 0 & -1 & 1 & 1 & 2 \end{pmatrix}.$$

Use the Simplex Algorithm to solve the LP and to find a certificate, as outlined in Problem 2, above.

Solution: First we modify A and b by multiplying rows 1 and 2 by -1, then we apply Phase I of the Simplex Algorithm, indicating pivot positions in bold.

$$\begin{pmatrix} 0 & u^T & 0 \\ A & I & b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -2 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & -3 & 2 & 0 & 1 & 0 & 2 \\ -2 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 & 3 & 4 & -4 & 0 & 0 & 0 & -5 \\ -1 & -1 & -1 & -2 & \mathbf{1} & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & -3 & 2 & 0 & 1 & 0 & 2 \\ -2 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & -2 & -1 & -4 & 0 & 4 & 0 & 0 & -1 \\ -1 & -1 & -1 & -2 & 1 & 1 & 0 & 0 & 1 \\ \mathbf{2} & 1 & 1 & 1 & 0 & -2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 3 & 0 & -1 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & -\frac{3}{2} & -\frac{1}{2} & -\frac{7}{2} & 0 & 3 & \frac{1}{2} & 0 & -1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 & 0 & \frac{1}{2} & 1 & 1 \\ 1 & \mathbf{\frac{1}{2}} & \frac{1}{2} & \frac{1}{2} & 0 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & \frac{7}{2} & 0 & -2 & \frac{1}{2} & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 3 & 0 & 1 & -2 & 0 & 0 & 2 & 0 & -1 \\ 1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 1 \\ 2 & 1 & 1 & \mathbf{1} & 0 & -2 & 1 & 0 & 0 \\ -3 & 0 & -1 & 2 & 0 & 1 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 7 & 2 & 3 & 0 & 0 & -4 & 4 & 0 & -1 \\ 3 & 1 & 1 & 0 & 1 & -3 & 2 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 & -2 & 1 & 0 & 0 \\ -7 & -2 & -3 & 0 & 0 & \mathbf{5} & -3 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} \frac{7}{5} & \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & \frac{8}{5} & \frac{4}{5} & -\frac{1}{5} \\ -\frac{6}{5} & -\frac{1}{5} & -\frac{4}{5} & 0 & 1 & 0 & \frac{1}{5} & \frac{3}{5} & \frac{8}{5} \\ -\frac{4}{5} & \frac{1}{5} & -\frac{1}{5} & 1 & 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ -\frac{7}{5} & -\frac{2}{5} & -\frac{3}{5} & 0 & 0 & 1 & -\frac{3}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}.$$

Since $\widetilde{c} \leq 0$, we see that the maximum value of w has been attained. The maximum value is $w_{\max} = -\frac{1}{5}$, and it occurs at the basic point $(\overline{x}, \overline{s})$ for the basis $B = \{4, 5, 6\}$. Since $w_{\max} < 0$, we know that the given LP is not feasible. By Problem 2(a), a certificate of optimality for $(\overline{x}, \overline{s})$ for the Auxiliary LP is given by the vector

$$y = \big(A\ I\big)_B^{-T} \begin{pmatrix} 0 \\ -u \end{pmatrix}_B = \begin{pmatrix} -2 & 1 & 1 \\ -3 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{-T} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

We have

$$\left(\begin{array}{ccc|c} -2 & -3 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & -3 & 1 & -2 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 0 & \frac{5}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 0 & 1 & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{3}{5} \\ 1 & 0 & 0 & -1 \end{array}\right).$$

Thus $y = \big(-1, \frac{3}{5}, -\frac{1}{5}\big)^T$ is a certificate of optimality for the basic point $(\overline{x}, \overline{s})$ for the Auxiliary LP. By Problem 2(c), the same vector y is a certificate of unfeasibility for the original LP.

4: Consider the LP in SEF with the following tableau

$$\begin{pmatrix} -1 & -2 & -2 & 1 & 3 & 1 \\ 1 & 1 & 1 & -2 & -1 & 4 \\ 1 & 1 & 0 & 1 & -2 & 3 \\ -1 & 0 & 1 & -5 & 2 & -3 \end{pmatrix}.$$

Use the Simplex Algorithm to solve the LP and to find a certificate, as outlined in Problem 2.

Solution: We multiply the bottom row of the tableau by -1, then apply Phase I of the Simplex Algorithm. We have

$$\begin{pmatrix} 0 & u^T & 0 \\ A & I & b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -2 & -1 & 1 & 0 & 0 & 4 \\ 1 & 1 & 0 & 1 & -2 & 0 & 1 & 0 & 3 \\ 1 & 0 & -1 & 5 & -2 & 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} -3 & -2 & 0 & -4 & 5 & 0 & 0 & 0 & -10 \\ 1 & 1 & 1 & -2 & -1 & 1 & 0 & 0 & 4 \\ 1 & 1 & 0 & 1 & -2 & 0 & 1 & 0 & 3 \\ 1 & 0 & -1 & 5 & -2 & 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 1 & 0 & -1 & -1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & \mathbf{1} & 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 1 & -2 & 0 & 1 & 0 & 3 \\ 0 & -1 & -1 & 4 & 0 & 0 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 & -4 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & -5 & 0 & 2 & -1 & 0 & 5 \\ 0 & -1 & -1 & \mathbf{4} & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -\frac{3}{4} & \frac{1}{4} & 0 & 1 & 1 & -\frac{7}{4} & \frac{3}{4} & 1 \\ 1 & -\frac{1}{4} & \frac{3}{4} & 0 & 0 & 2 & -\frac{9}{4} & \frac{5}{4} & 5 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}.$$

Phase I ends with $w_{\max} = 0$ and this occurs at the basic point for the basis $B = \{1, 4, 5\}$. We then apply Phase II of the Simplex Algorithm. We begin by putting the tableau into canonical form for the feasible basis $B = \{1, 4, 5\}$. We have

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} -1 & -2 & -2 & 1 & 3 & 1 \\ 1 & 1 & 1 & -2 & -1 & 4 \\ 1 & 1 & 0 & 1 & -2 & 3 \\ 1 & 0 & -1 & 5 & -2 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & -1 & -1 & 2 & 5 \\ 1 & 1 & 1 & -2 & -1 & 4 \\ 0 & 0 & 1 & -3 & 1 & 1 \\ 0 & 1 & 2 & -7 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & -3 & -5 & 13 & 0 & 3 \\ 1 & 2 & 3 & -9 & 0 & 5 \\ 0 & -1 & -1 & 4 & 0 & 0 \\ 0 & 1 & 2 & -7 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & \frac{1}{4} & -\frac{7}{4} & 0 & 0 & 3 \\ 1 & -\frac{1}{4} & \frac{3}{4} & 0 & 0 & 5 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & 0 & 0 \\ 0 & -\frac{3}{4} & \frac{1}{4} & 0 & 1 & 1 \end{pmatrix}.$$

Then we perform iterations of the Simplex Algorithm. We have

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} 0 & \frac{1}{4} & -\frac{7}{4} & 0 & 0 & 3 \\ 1 & -\frac{1}{4} & \frac{3}{4} & 0 & 0 & 5 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & 0 & 0 \\ 0 & -\frac{3}{4} & \mathbf{\frac{1}{4}} & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & -5 & 0 & 0 & 7 & 10 \\ 1 & \mathbf{2} & 0 & 0 & -3 & 2 \\ 0 & -1 & 0 & 1 & 1 & 1 \\ 0 & -3 & 1 & 0 & 4 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} \frac{5}{2} & 0 & 0 & 0 & -\frac{1}{2} & 15 \\ \frac{1}{2} & 1 & 0 & 0 & -\frac{3}{2} & 1 \\ \frac{1}{2} & 0 & 0 & 1 & -\frac{1}{2} & 2 \\ \frac{3}{2} & 0 & 1 & 0 & -\frac{1}{2} & 7 \end{pmatrix} = \begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}.$$

We see that $\widetilde{c}_5 < 0$ and $\widetilde{A}_5 \leq 0$ and so the original LP is unbounded, and by Problem 2(b) a certificate of unboundedness is given by the basic point $\overline{x} = \left(0, 1, 7, 2, 0\right)^T$ for the basis $B = \{2, 3, 4\}$ together with the vector $y = \left(0, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 1\right)^T$.

5: Consider the LP in SEF with the following tableau

$$\begin{pmatrix} -1 & 1 & -1 & -1 & -3 & -2 \\ 1 & -1 & 2 & 1 & 1 & 4 \\ 1 & 1 & -1 & 2 & 4 & 5 \\ -1 & 2 & -1 & 2 & 3 & 4 \end{pmatrix}.$$

Use the Simplex Algorithm to solve the LP and to find a certificate, as outlined in Problem 2.

Solution: First we apply Phase I of the Simplex Algorithm. We have

$$\begin{pmatrix} 0 & u^T & 0 \\ A & I & b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 1 & 1 & 1 & 0 & 0 & 4 \\ 1 & 1 & -1 & 2 & 4 & 0 & 1 & 0 & 5 \\ -1 & 2 & -1 & 2 & 3 & 0 & 0 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} -1 & -2 & 0 & -5 & -8 & 0 & 0 & 0 & -13 \\ 1 & -1 & 2 & 1 & 1 & 1 & 0 & 0 & 4 \\ 1 & 1 & -1 & 2 & 4 & 0 & 1 & 0 & 5 \\ -1 & 2 & -1 & 2 & 3 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & -3 & 2 & -4 & -7 & 1 & 0 & 0 & -9 \\ 1 & -1 & 2 & 1 & 1 & 1 & 0 & 0 & 4 \\ 0 & \mathbf{2} & -3 & 1 & 3 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 & 4 & 1 & 0 & 1 & 8 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{1}{2} & \frac{3}{2} & 0 & -\frac{15}{2} \\ 1 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{9}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \mathbf{\frac{5}{2}} & \frac{5}{2} & \frac{5}{2} & \frac{3}{2} & -\frac{1}{2} & 1 & \frac{15}{2} \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 2 & \frac{1}{5} & \frac{3}{5} & -\frac{1}{5} & 3 \\ 0 & 1 & 0 & 2 & 3 & \frac{2}{5} & \frac{1}{5} & \frac{3}{5} & 5 \\ 0 & 0 & 1 & 1 & 1 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & 3 \end{pmatrix}.$$

Phase I ends with $w_{\max} = 0$ and this occurs at the basic point for the basis $B = \{1, 2, 3\}$. We then apply Phase II, first putting the tableau into near-canonical form for the basis $B = \{1, 2, 3\}$. We have

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 & -1 & -3 & -2 \\ 1 & -1 & 2 & 1 & 1 & 4 \\ 1 & 1 & -1 & 2 & 4 & 5 \\ -1 & 2 & -1 & 2 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 & 0 & -2 & 2 \\ 1 & -1 & 2 & 1 & 1 & 4 \\ 0 & 2 & -3 & 1 & 3 & 1 \\ 0 & 1 & 1 & 3 & 4 & 8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 1 & 0 & -2 & 2 \\ 1 & 0 & 3 & 4 & 5 & 12 \\ 0 & 0 & -5 & -5 & -5 & -15 \\ 0 & 1 & 1 & 3 & 4 & 8 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & -1 & -3 & -1 \\ 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 3 & 5 \end{pmatrix}.$$

Then we perform iterations of the Simplex Algorithm. We have

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & -1 & -3 & -1 \\ 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & -1 & -3 & -1 \\ 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & \mathbf{2} & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & -\frac{3}{2} & \frac{3}{2} \\ 1 & -\frac{1}{2} & 0 & 0 & \mathbf{\frac{1}{2}} & \frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 1 & \frac{3}{2} & \frac{5}{2} \end{pmatrix}$$

$$\sim \begin{pmatrix} 3 & -1 & 0 & 0 & 0 & 3 \\ 2 & -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ -3 & \mathbf{2} & 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} \frac{3}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{7}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 & \frac{3}{2} \\ -\frac{3}{2} & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}.$$

We find that $z_{\max} = \frac{7}{2}$ and this occurs at the basic point $\overline{x} = \left(0, \frac{1}{2}, \frac{3}{2}, 0, \frac{3}{2}\right)^T$ for the basis $B = \{2, 3, 5\}$. By Problem 2(a), a certificate of optimality is given by the vector $y = {A_B}^{-T} c_B$. We have

$$\left({A_B}^T \middle| c_B\right) = \left(\begin{array}{ccc|c} -1 & 1 & 2 & -1 \\ 2 & -1 & -1 & 1 \\ 1 & 4 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 5 & 5 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -10 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{7}{10} \\ 0 & 1 & 0 & \frac{11}{10} \\ 0 & 0 & 1 & -\frac{7}{10} \end{array}\right)$$

Thus $y = \left(\frac{7}{10}, \frac{11}{10}, -\frac{7}{10}\right)^T$ is a certificate of optimality for the optimal solution $\overline{x} = \left(0, \frac{1}{2}, \frac{3}{2}, 0, \frac{3}{2}\right)^T$.