CO 250 Intro to Optimization, Solutions to Assignment 3

1: Consider an LP with constraints $Ax = b$ and $x \geq 0$ where

$$A = \begin{pmatrix} 1 & 1 & 2 & -1 & -4 & 0 \\ 1 & 0 & -1 & 1 & 1 & -2 \\ 2 & 1 & 3 & -1 & -4 & -3 \\ 1 & 2 & 2 & 0 & -3 & 2 \end{pmatrix} \;,\; b = \begin{pmatrix} -1 \\ 2 \\ -1 \\ 8 \end{pmatrix} .$$

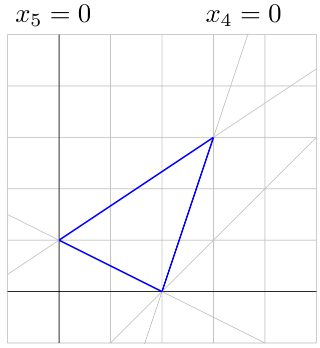
Use a picture of the feasible set to find every feasible basis for the LP and all of the corresponding feasible basic points.

Solution: We solve $Ax = b$. We have

$$(A|b) = \left(\begin{array}{cccccc|c} 1 & 1 & 2 & -1 & -4 & 0 & -1 \\ 1 & 0 & -1 & 1 & 1 & -2 & 2 \\ 2 & 1 & 3 & -1 & -4 & -3 & -1 \\ 1 & 2 & 2 & 0 & -3 & 2 & 8 \end{array}\right) \sim \left(\begin{array}{cccccc|c} 1 & 1 & 2 & -1 & -4 & 0 & -1 \\ 0 & 1 & 3 & -2 & -5 & 2 & -3 \\ 0 & 1 & 1 & -1 & -4 & 3 & -1 \\ 0 & 1 & 0 & 1 & 1 & 2 & 9 \end{array}\right) \sim \left(\begin{array}{cccccc|c} 1 & 0 & -1 & 1 & 1 & -2 & 2 \\ 0 & 1 & 3 & -2 & -5 & 2 & -3 \\ 0 & 0 & 2 & -1 & -1 & -1 & -2 \\ 0 & 0 & 3 & -3 & -6 & 0 & -12 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & -2 & -2 \\ 0 & 1 & 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 1 & -1 & -2 & 0 & -4 \\ 0 & 0 & 0 & 1 & 3 & -1 & 6 \end{array}\right) \sim \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & -2 & -2 \\ 0 & 1 & 0 & 0 & -2 & 3 & 3 \\ 0 & 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -1 & 6 \end{array}\right) .$$

The solution to $Ax = b$ is given by $x = p + su + tv$ where $p = (-2, 3, 2, 6, 0, 0)^T$, $u = (1, 2, -1, -3, 1, 0)^T$ and $v = (2, -3, 1, 1, 0, 1)^T$. The constraint $x \geq 0$ becomes $s + 2t \geq 2$, $2s - 3t \geq -3$, $-s + t \geq -2$, $-3s + t \geq -6$, $s \geq 0$ and $t \geq 0$. We draw a picture of the feasible set, outlined in blue, along with the lines given by $x_i = 0$.



The vertices of the feasible set (that is the feasible basic points) are given by $(s, t) = (0, 1)$, $(2, 0)$ and $(3, 3)$. The point $(s, t) = (3, 3)$ occurs when $x_2 = x_4 = 0$, and it gives the vertex $\overline{x} = p + 3u + 3v = (7, 0, 2, 0, 3, 3)^T$ which is the basic point for the basis

$$B = \{1, 3, 5, 6\} .$$

The point $(s, t) = (0, 1)$ occurs when $x_1 = x_2 = x_5 = 0$, and it gives the vertex $\overline{x} = p + t = (0, 0, 3, 7, 0, 1)^T$ which is the basis point for any of the 3 bases

$$B = \{1, 3, 4, 6\}, \{2, 3, 4, 6\}, \{3, 4, 5, 6\} .$$

The point $(s, t) = (2, 0)$ occurs when $xx_1 = x_3 = x_4 = x_6 = 0$, and it gives the vertex $\overline{x} = (0, 7, 0, 0, 2, 0)^T$ which is the basic point for any one of the 6 bases

$$B = \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 2, 5, 6\}, \{2, 3, 4, 5\}, \{2, 3, 5, 6\}, \{2, 4, 5, 6\} .$$

2: Consider the LP where we maximize $z(x) = c_0 + c^T x$ subject to $Ax = b$, $x \geq 0$ where

$$c_0 = 6\ ,\ c = \big(3, 1, -2, 1, 2, -5\big)^T\ ,\ A = \begin{pmatrix} 2 & 1 & -1 & 2 & 1 & 0 \\ 4 & 2 & -2 & 5 & 3 & 1 \\ 3 & 1 & -2 & 4 & 4 & -1 \end{pmatrix}\ ,\ b = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

Put the LP into canonical form for the basis $B = \{2, 4, 5\}$ in the following two ways.

(a) Use row operations on the tableau $\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix}$ to obtain the tableau $\begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}$.

Solution: We have

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} -3 & -1 & 2 & -1 & -2 & 5 & 6 \\ 2 & 1 & -1 & 2 & 1 & 0 & 1 \\ 4 & 2 & -2 & 5 & 3 & 1 & 3 \\ 3 & 1 & -2 & 4 & 4 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 & 1 & -1 & 5 & 7 \\ 2 & 1 & -1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 2 & 3 & -1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 0 & 1 & 0 & -2 & 4 & 6 \\ 2 & 1 & -1 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 1 & -3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & -2 & 4 \\ 3 & 1 & -2 & 0 & 0 & -5 & -2 \\ -1 & 0 & 1 & 1 & 0 & 4 & 2 \\ 1 & 0 & -1 & 0 & 1 & -3 & -1 \end{pmatrix} = \begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}.$$

(b) Calculate ${A_B}^{-1}$ then use the formulas $\widetilde{A} = {A_B}^{-1} A$, $\widetilde{b} = {A_B}^{-1} b$ and $\widetilde{c}_0 = c_0 + b^T y$ and $\widetilde{c} = c - A^T y$ where $y = {A_B}^{-T} c_B$.

Solution: We calculate ${A_B}^{-1}$. We have

$$\big(A_B \big| I\big) = \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 4 & 4 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -4 & 1 \\ 0 & 1 & 0 & -5 & 3 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right)$$

and so

$${A_B}^{-1} = \begin{pmatrix} 8 & -4 & 1 \\ -5 & 3 & -1 \\ 3 & -2 & 1 \end{pmatrix}.$$

Thus

$$\widetilde{A} = {A_B}^{-1} A = \begin{pmatrix} 8 & -4 & 1 \\ -5 & 3 & -1 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 & 2 & 1 & 0 \\ 4 & 2 & -2 & 5 & 3 & 1 \\ 3 & 1 & -2 & 4 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -2 & 0 & 0 & -5 \\ -1 & 0 & 1 & 1 & 0 & 4 \\ 1 & 0 & -1 & 0 & 1 & -3 \end{pmatrix}$$

$$\widetilde{b} = {A_B}^{-1}\, b = \begin{pmatrix} 8 & -4 & 1 \\ -5 & 3 & -1 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$$

$$y = {A_B}^{-T} c_B = \begin{pmatrix} 8 & -5 & 3 \\ -4 & 3 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ -5 \\ 2 \end{pmatrix}$$

$$\widetilde{c}_0 = c_0 + b^T y = 6 + (1, 3, 2) \cdot (9, -5, 2) = 4$$

$$\widetilde{c} = c - A^T y = \begin{pmatrix} 3 \\ 1 \\ -2 \\ 1 \\ 2 \\ -5 \end{pmatrix} - \begin{pmatrix} 2 & 4 & 3 \\ 1 & 2 & 1 \\ -1 & -2 & -2 \\ 2 & 5 & 4 \\ 1 & 3 & 4 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \\ 1 \\ 2 \\ -5 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ -3 \\ 1 \\ 2 \\ -7 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}.$$

3: Consider the LP where we maximize $z(x) = c_0 + c^T x$ subject to $Ax = b$, $x \geq 0$, where

$$c_0 = 2\ ,\ c = \big(1, 1, -1, 1, -5\big)^T\ ,\ A = \begin{pmatrix} 1 & 2 & -1 & -4 & -2 \\ 2 & 1 & -1 & -1 & -6 \\ -1 & 1 & 1 & -2 & 5 \end{pmatrix}\ ,\ b = \begin{pmatrix} -4 \\ -9 \\ 10 \end{pmatrix}.$$

Let $B = \{2, 3, 5\}$, $B' = \{2, 4, 5\}$ and $B'' = \{1, 2, 4\}$. By performing row operations on the tableau, find the basic points u, u', u'' for these bases, find the values $z(u)$, $z(u')$ and $z(u'')$, and find an optimal solution to the LP.

Solution: First we put the tableau in (near) canonical form for the basis $B = \{2, 3, 5\}$. We have

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 & -1 & 5 & 2 \\ 1 & 2 & -1 & -4 & -2 & -4 \\ 2 & 1 & -1 & -1 & -6 & -9 \\ -1 & 1 & 1 & -2 & 5 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -2 & -1 & -7 \\ -3 & 0 & 1 & -2 & 10 & 14 \\ 2 & 1 & -1 & -1 & -6 & -9 \\ -3 & 0 & 2 & -1 & 11 & 19 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -2 & -1 & -7 \\ -3 & 0 & 1 & -2 & 10 & 14 \\ -1 & 1 & 0 & -3 & 4 & 5 \\ 3 & 0 & 0 & 3 & -9 & -9 \end{pmatrix} \sim \begin{pmatrix} \frac{2}{3} & 0 & 0 & -\frac{7}{3} & 0 & -6 \\ \frac{1}{3} & 0 & 1 & \frac{4}{3} & 0 & 4 \\ \frac{1}{3} & 1 & 0 & -\frac{5}{3} & 0 & 1 \\ -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 1 \end{pmatrix}$$

Thus the basic point for $B = \{2, 3, 5\}$ is $u = (0, 1, 4, 0, 1)^T$ and we have $z(u) = -6$. Next we pivot at position $(4, 1)$ to put the tableau into (near) canonical form for $B' = \{2, 4, 5\}$. We obtain

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} \frac{5}{4} & 0 & \frac{7}{4} & 0 & 0 & 1 \\ \frac{1}{4} & 0 & \frac{3}{4} & 1 & 0 & 3 \\ \frac{3}{4} & 1 & \frac{5}{4} & 0 & 0 & 6 \\ -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 1 & 2 \end{pmatrix}.$$

Thus the basic point for B' is $u' = (0, 6, 0, 3, 2)$ and we have $z(u') = 1$. Also notice that u' is an optimal solution to the LP as we can see from row 0 in the above tableau. Finally, we pivot at position $(3, 1)$ to put the tableau into (near) canonical form for the basis $B'' = \{1, 2, 4\}$.

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 3 & 0 & 5 & 11 \\ 0 & 0 & 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 0 & 3 & 12 \\ 1 & 0 & -1 & 0 & -4 & -8 \end{pmatrix}.$$

Thus the basic point for B'' is $u'' = (-8, 12, 0, 5, 0)^T$ and we have $z(u'') = 11$. Note that, although we have $z(u'') > z(u')$, u'' is not an optimal solution to the LP because it is not a feasible point.

4: Consider the LP where we maximize $z(x) = c_0 + c^T x$ subject to $Ax = b$, $x \geq 0$, where

$$c_0 = 3\ ,\ c = \big(1, -2, 1, 3, -1\big)^T\ ,\ A = \begin{pmatrix} 1 & 2 & 1 & 3 & -4 \\ 2 & 1 & 1 & 0 & -4 \\ 1 & -3 & 1 & -2 & 1 \end{pmatrix}\ ,\ b = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}.$$

Use Phase II of the Simplex Algorithm, starting with the feasible basis $B = \{2,3,5\}$, to solve the LP.

Solution: First we put the tableau into (near) canonical form for the basis $B = \{2,3,5\}$. We have

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 & -3 & 1 & 3 \\ 1 & 2 & 1 & 3 & -4 & 5 \\ 2 & 1 & 1 & 0 & -4 & 2 \\ 1 & -3 & 1 & -2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} -5 & 0 & -3 & -3 & 9 & -1 \\ -3 & 0 & -1 & 3 & 4 & 1 \\ 2 & 1 & 1 & 0 & -4 & 2 \\ 7 & 0 & 4 & -2 & -11 & 6 \end{pmatrix}$$
$$\sim \begin{pmatrix} 4 & 0 & 0 & -12 & -3 & -4 \\ 3 & 0 & 1 & -3 & -4 & -1 \\ -1 & 1 & 0 & 3 & 0 & 3 \\ -5 & 0 & 0 & 10 & 5 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -6 & 0 & 2 \\ -1 & 0 & 1 & 5 & 0 & 7 \\ -1 & 1 & 0 & 3 & 0 & 3 \\ -1 & 0 & 0 & 2 & 1 & 2 \end{pmatrix}.$$

Next we perform iterations of the Simplex Algorithm, showing the pivot positions in bold.

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -6 & 0 & 2 \\ -1 & 0 & 1 & 5 & 0 & 7 \\ -1 & 1 & 0 & \mathbf{3} & 0 & 3 \\ -1 & 0 & 0 & 2 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 0 & 0 & 0 & 8 \\ \mathbf{\frac{2}{3}} & -\frac{5}{3} & 1 & 0 & 0 & 2 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 1 & 0 & 1 \\ -\frac{1}{3} & -\frac{2}{3} & 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{2} & 0 & 0 & 11 \\ 1 & -\frac{5}{2} & \frac{3}{2} & 0 & 0 & 3 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 1 & 0 & 2 \\ 0 & -\frac{3}{2} & \frac{1}{2} & 0 & 1 & 1 \end{pmatrix}.$$

From the second column, we see that the LP is unbounded. Indeed, for the equivalent LP with the final tableau above, a certificate of unboundedness is given by the basic point $\overline{x} = (3,0,0,2,1)^T$ together with the vector $y = \big(\frac{5}{2}, 1, 0, \frac{1}{2}, \frac{3}{2}\big)^T$.

5: Consider the LP where we maximize $z(x) = c_0 + c^T x$ subject to $Ax = b$, $x \geq 0$, where

$$c_0 = -2\ ,\ c = \big(-2, -4, -1, 1, 4, 3\big)^T\ ,\ A = \begin{pmatrix} 1 & 5 & 2 & -1 & -1 & 0 \\ 2 & 0 & -3 & 1 & -3 & -1 \\ 3 & 4 & -1 & 1 & -2 & 1 \end{pmatrix}\ ,\ b = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}.$$

Use Phase II of the Simplex Algorithm, starting with the feasible basis $B = \{1, 3, 4\}$, to solve the LP.

Solution: First we put the tableau into (near) canonical form for the basis $B = \{1, 3, 4\}$. We have

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 & -1 & -4 & -3 & -2 \\ 1 & 5 & 2 & -1 & -1 & 0 & -1 \\ 2 & 0 & -3 & 1 & -3 & -1 & -1 \\ 3 & 4 & -1 & 1 & -2 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & -6 & -3 & 1 & -2 & -3 & 0 \\ 1 & 5 & 2 & -1 & -1 & 0 & -1 \\ 0 & 10 & 7 & -3 & 1 & 1 & -1 \\ 0 & 11 & 7 & -4 & -1 & -1 & -6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & -6 & -3 & 1 & -2 & -3 & 0 \\ 1 & 5 & 2 & -1 & -1 & 0 & -1 \\ 0 & 10 & 7 & -3 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 & 2 & 2 & 5 \end{pmatrix} \sim \begin{pmatrix} 0 & -5 & -3 & 0 & -4 & -5 & -5 \\ 1 & 4 & 2 & 0 & 1 & 2 & 4 \\ 0 & 7 & 7 & 0 & 7 & 7 & 14 \\ 0 & -1 & 0 & 1 & 2 & 2 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & -5 & -3 & 0 & -4 & -5 & -5 \\ 1 & 4 & 2 & 0 & 1 & 2 & 4 \\ 0 & 1 & 1 & 0 & 1 & 1 & 2 \\ 0 & -1 & 0 & 1 & 2 & 2 & 5 \end{pmatrix} \sim \begin{pmatrix} 0 & -2 & 0 & 0 & -1 & -2 & 1 \\ 1 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 2 \\ 0 & -1 & 0 & 1 & 2 & 2 & 5 \end{pmatrix}.$$

Next we perform iterations of the Simplex Algorithm, indicating pivot positions in bold.

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} 0 & -2 & 0 & 0 & -1 & -2 & 1 \\ 1 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \mathbf{1} & 1 & 2 \\ 0 & -1 & 0 & 1 & 2 & 2 & 5 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & -1 & 3 \\ 1 & \mathbf{3} & 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 & 2 \\ 0 & -3 & -2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} \frac{1}{3} & 0 & \frac{4}{3} & 0 & 0 & -\frac{2}{3} & \frac{11}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 & \boldsymbol{\frac{1}{3}} & \frac{2}{3} \\ -\frac{1}{3} & 0 & \frac{2}{3} & 0 & 1 & \frac{2}{3} & \frac{4}{3} \\ 1 & 0 & -1 & 1 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 2 & 0 & 0 & 0 & 5 \\ 1 & 3 & 1 & 0 & 0 & 1 & 2 \\ -1 & -2 & 0 & 0 & 1 & 0 & 0 \\ -2 & -3 & -2 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

From row 0 we see that we have attained the optimal solution. The maximum value for z is $z_{\max} = 5$, and it occurs at the point $\overline{x} = \big(0, 0, 0, 1, 0, 2\big)^T$. Indeed, for the equivalent LP with the final tableau, the vector $y = 0$ is a certificate of optimality for the point $\overline{x}$.