CO 250 Intro to Optimization, Solutions to Assignment 2

1: (a) Consider the LP where we *minimize* $z(x) = 3 - 2x_1 - x_2 + 2x_3$ subject to the constraints

$$-x_1 - 2x_2 + 3x_3 = 1\ ,\ 3x_1 + x_2 - x_3 = 2$$
$$-2x_1 + 3x_2 - 2x_3 \le 4\ ,\ -x_1 - x_2 + 2x_3 \ge 3\ ,\ x_1 \le 0\,.$$

Convert this to an equivalent LP in SEF for $\widetilde{x} = \left(x_1{}^-, x_2{}^+, x_2{}^-, x_3{}^+, x_3{}^-, s\,, t\,\right)^T \in \mathbf{R}^7$. Express the answer in matrix form (that is in the form where we maximize $\widetilde{z}(\widetilde{x}) = \widetilde{c}_0 + \widetilde{c}^T$ subject to $\widetilde{A}\widetilde{x} = \widetilde{b}$, $\widetilde{x} \ge 0$).

Solution: Note that we must maximize $\widetilde{z} = -z = -3 + 2x_1 + x_2 - 2x_3$. We introduce the suggested variables $x_1{}^-, x_2{}^+, x_2{}^-, x_3{}^+, x_3{}^-, s\,, t$, and write $x_1 = -x_1{}^-$, $x_2 = x_2{}^+ - x_2{}^-$ and $x_3 = x_3{}^+ - x_3{}^-$, and we use s and t as slack variables. In terms of these new variables, we maximize

$$\widetilde{z} = -3 - 2x_1{}^- + x_2{}^+ - x_2{}^- - 2x_3{}^+ + 2x_3{}^-$$

subject to

$$x_1{}^- - 2x_2{}^+ + 2x_2{}^- + 3x_3{}^+ - 3x_3{}^- = 1\ \ ,\ \ -3x_1{}^- + x_2{}^+ - x_2{}^- - x_3{}^+ + x_3{}^- = 2\ ,$$
$$2x_1{}^- + 3x_2{}^+ - 3x_2{}^- - 2x_3{}^+ + 2x_3{}^- + s = 4\ \ ,\ \ x_1{}^- - x_2{}^+ + x_2{}^- + 2x_3{}^+ - 2x_3{}^- - t = 3$$
$$x_1{}^- \ge 0\ ,\ x_2{}^+ \ge 0\ ,\ x_2{}^- \ge 0\ ,\ x_3{}^+ \ge 0\ ,\ x_3{}^- \ge 0\ ,\ s \ge 0\ ,\ t \ge 0\,.$$

In matrix form, we maximize $\widetilde{z}(\widetilde{x}) = \widetilde{c}_0 + \widetilde{c}^T$ subject to $\widetilde{A}\widetilde{x} = \widetilde{b}$, $\widetilde{x} \ge 0$ where

$$\widetilde{c}_0 = -3\ ,\ \widetilde{c} = \big(-2, 1, -1, -2, 2, 0, 0\big)^T\ ,\ \widetilde{A} = \begin{pmatrix} 1 & -2 & 2 & 3 & -3 & 0 & 0 \\ -3 & 1 & -1 & -1 & 1 & 0 & 0 \\ 2 & 3 & -3 & -2 & 2 & 1 & 0 \\ 1 & -1 & 1 & 2 & -2 & 0 & -1 \end{pmatrix}\ ,\ \widetilde{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}.$$

(b) Consider the LP where we maximize $z(x) = 3x_1 - x_2 + 2x_3 + x_4 - 2x_5$ subject to the constraints

$$x_1 + 2x_2 + x_3 + 3x_4 - x_5 = 3\ ,\ 2x_1 + 3x_2 + x_3 + 4x_4 = 5$$
$$3x_1 + 2x_2 + x_3 - x_4 + 6x_5 \le 4\ ,\ x_1 \ge 0\ ,\ x_3 \ge 0\,.$$

Solve the equality constraints for x_2 and x_4 in terms of x_1, x_3 and x_5, and then convert this LP to an equivalent LP in SEF for $\widetilde{x} = \left(x_1, x_3, x_5{}^+, x_5{}^-, s\,\right)^T \in \mathbf{R}^5$. Express the answer in matrix form.

Solution: First we solve the two equality constraints reducing the associated augmented matrix putting pivots in columns 2 and 4 (in the first step we replace row 1 by row 2-row 1).

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & -1 & 3 \\ 2 & 3 & 1 & 4 & 0 & 5 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 1 & 2 \\ 2 & 3 & 1 & 4 & 0 & 5 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 1 & 2 \\ -1 & 0 & 1 & 1 & -3 & -1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 2 & 1 & -1 & 0 & 4 & 3 \\ -1 & 0 & 1 & 1 & -3 & -1 \end{array}\right)$$

The solution to the two equality constraints is given by $x_2 = 3 - 2x_1 + x_3 - 4x_5$ and $x_4 = -1 + x_1 - x_3 + 3x_5$. We put these expressions for x_2 and x_4 into the formula for z to get

$$z = 3x_1 - (3 - 2x_1 + x_3 - 4x_5) + 2x_3 + (-1 + x_1 - x_3 + 3x_5) - 2x_5 = -4 + 6x_1 + 0\,x_3 + 5x_5$$

and we put them into the inequality constraint to get

$$3x_1 + 2(3 - 2x_1 + x_3 - 4x_5) + x_3 - (-1 + x_1 - x_3 + 3x_5) + 6x_5 \le 4$$

that is

$$-2x_1 + 4x_3 - 5x_5 \le -3\,.$$

We introduce variables $x_5{}^+$ and $x_5{}^-$ and set $x_5 = x_5{}^+ - x_5{}^-$, and we introduce the slack variable s. The given LP is equivalent to the LP where we maximize $z = -4 + 6x_1 + 0\,x_3 + 5x_5{}^+ - 5x_5{}^-$ subject to $-2x_1 + 4x_3 - 5x_5{}^+ + 5x_5{}^- + s = -3$. In matrix form, we maximize $\widetilde{z} = z = \widetilde{c}_0 + \widetilde{c}^T\widetilde{x}$ for $\widetilde{x} \in \mathbf{R}^5$ subject to $\widetilde{A}\widetilde{x} = \widetilde{b}$, $\widetilde{x} \ge 0$ where $\widetilde{x} = \left(x_1, x_3, x_5{}^+, x_5{}^-, s\,\right)^T$ and

$$\widetilde{c}_0 = -4\ ,\ \widetilde{c} = \big(6, 0, 5, -5, 0\big)^T\ ,\ \widetilde{A} = \begin{pmatrix} -2 & 4 & -5 & 5 & 1 \end{pmatrix}\ ,\ \widetilde{b} = \big(-3\big)\,.$$

2: An LP in **Standard Inequality Form** (or SIF) is an LP in which we maximize the value of $z(x) = c_0 + c^T x$ for $x \in \mathbf{R}^n$ subject to $Ax \le b$, where $c_0 \in \mathbf{R}$, $c \in \mathbf{R}^n$ and $A \in M_{k\times n}(\mathbf{R})$.

(a) Show that every LP is equivalent to an LP in SIF by converting the LP where we maximize $z(x) = c_0 + c^T x$ subject to $Ax = u$, $Bx \ge v$, $Cx \le w$ into an equivalent LP in SEF. Express the answer in matrix form.

Solution: The given constraints can be written as

$$Ax \le u \ , \ Ax \ge u \ , \ Bx \ge v \ , \ Cx \le w$$

or equivalently as

$$Ax \le u \ , \ (-A)x \le (-u) \ , \ (-B)x \le (-v) \ , \ Cx \le w \, .$$

Thus the given LP is equivalent to the LP in SIF where we maximize $z(x) = c_0 + c^T x$ subject to

$$\begin{pmatrix} A \\ -A \\ -B \\ C \end{pmatrix} x \le \begin{pmatrix} u \\ -u \\ -v \\ w \end{pmatrix} .$$

(b) Consider the LP in SIF where we maximize $z(x) = c_0 + c^T x$ subject to $Ax \le b$.

(i) Show that if $y \in \mathbf{R}^k$ with $A^T y = 0$, $y \ge 0$ and $b^T y < 0$, then the LP is unfeasible.

Solution: Suppose that $y \in \mathbf{R}^k$ with $A^T y = 0$, $y \ge 0$ and $b^T y < 0$. Suppose, for a contradiction, that the LP is feasible. Choose a feasible point $\overline{x} \in \mathbf{R}^n$, so we have $A\overline{x} \le b$. Since $A\overline{x} \le b$ and $y \ge 0$ we have $y^T A\overline{x} \le y^T b$, and hence, by taking the transpose on both sides, $\overline{x}^T A^T y \le b^T y$. Since $A^T y = 0$ this gives $b^T y \ge 0$, contradicting the fact that $b^T y < 0$.

(ii) Show that if $\overline{x} \in \mathbf{R}^n$ with $A\overline{x} \le b$ and $y \in \mathbf{R}^n$ with $Ay \le 0$ and $c^T y > 0$ then the LP is unbounded.

Solution: Suppose that $\overline{x} \in \mathbf{R}^n$ with $A\overline{x} \le b$ and that $y \in \mathbf{R}^n$ with $Ay \le 0$ and $c^T y > 0$. Consider any point of he form $\overline{x} + ty$ with $t \ge 0$. Since $A\overline{x} \le b$, $Ay \le 0$ and $t \ge 0$, we have $A(\overline{x} + ty) = A\overline{x} + t\,Ay \le b$, so the point $\overline{x} + ty$ is feasible. Also, we have $z(\overline{x} + ty) = c_0 + c^T\overline{x} + t\,c^T y \to \infty$ as $t \to \infty$ since $c^T y > 0$. Thus the LP is unbounded.

(iii) Show that if $\overline{x} \in \mathbf{R}^n$ with $A\overline{x} \le b$ and $y \in \mathbf{R}^k$ with $A^T y = c$, $y \ge 0$ and $b^T y = c^T\overline{x}$, then $\overline{x}$ is an optimal solution for the LP.

Solution: Suppose that $\overline{x} \in \mathbf{R}^n$ with $A\overline{x} \le b$ and $y \in \mathbf{R}^k$ with $A^T y = c$, $y \ge 0$ and $b^T y = c^T\overline{x}$. Let x be any feasible point for the LP, so we have $Ax \le b$. Then

$$\begin{aligned} z(\overline{x}) &= c_0 + c^T\overline{x} = c_0 + b^T y && \text{, since } b^T y = c^T\overline{x}, \\ &= c_0 + y^T b \ge c_0 + y^T Ax && \text{, since } y \ge 0 \text{ and } Ax \le b, \\ &= c_0 + (A^T y)x = c_0 + c^T x && \text{, since } A^T y = c, \\ &= z(x) \, . \end{aligned}$$

Thus $z(\overline{x}) \ge z(x)$ for every feasible point x, in other words $\overline{x}$ is an optimal solution for the LP.

3: Consider the LP where we maximize $z = c_0 + c^T x$ for $x \in \mathbf{R}^5$ subject to the constraints $Ax = b$ and $x \geq 0$, where

$$c_0 = 1\ ,\ c = \big(2, -3, 1, 4, -2\big)^T\ ,\ A = \begin{pmatrix} 1 & -1 & 1 & 0 & 1 \\ 1 & 2 & 1 & -3 & -2 \\ 2 & 1 & 1 & -1 & -4 \end{pmatrix}\ ,\ b = \begin{pmatrix} 3 \\ -6 \\ -1 \end{pmatrix}.$$

Determine whether the LP is unfeasible, unbounded, or has an optimal solution, and find a certificate.

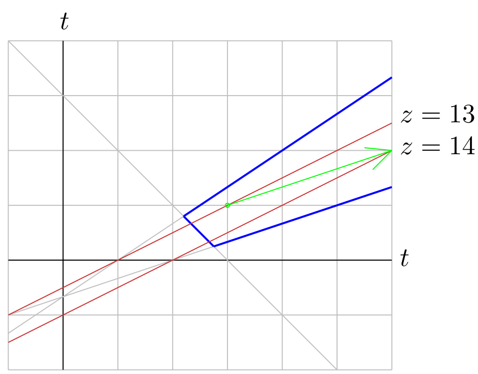
Solution: First we solve $Ax = b$. We have

$$\big(A\big|b\big) = \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 1 & 3 \\ 1 & 2 & 1 & -3 & -2 & -6 \\ 2 & 1 & 1 & -1 & -4 & -1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 1 & 3 \\ 0 & 3 & 0 & -3 & -3 & -9 \\ 0 & 3 & -1 & -1 & -6 & -7 \end{array}\right)$$
$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -3 \\ 0 & 0 & 1 & -2 & 3 & -2 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -1 & -1 & -3 \\ 0 & 0 & 1 & -2 & 3 & -2 \end{array}\right).$$

The solution to $Ax = b$ is given by $x = p + su + tv$ where $p = \big(2, -3, -2, 0, 0\big)^T$, $u = \big(-1, 1, 2, 1, 0\big)^T$ and $v = \big(3, 1, -3, 0, 1\big)^T$. We must maximize

$$z = c_0 + c^T x = (c_0 + c^T p) + (c^T u)s + (c^T v)t = 12 + s - 2t$$

subject to the constraints $x_1 \geq 0, x_2 \geq 0, \cdots, x_5 \geq 0$ which we rewrite as $-s+3t \geq -2$, $s+t \geq 3$, $2s-3t \geq 2$, $s \geq 0$ and $t \geq 0$. We draw a picture of the set of points (s,t) which satisfy these constraints (outlined in blue) along with some of the level curves $z =$ constant (shown in orange).



We see that the LP is unbounded. One feasible point is given by $(s,t) = (3,1)$, that is the point

$$\overline{x} = p + 3u + v = (2, 1, 1, 3, 1)\,.$$

From the point given by $(s,t) = (3,1)$, we can move in the direction of the vector $(s,t) = (3,1)$ and remain in the feasible set with the value of z increasing arbitrarily high, so a certificate of unfeasibility is given by the vector

$$y = 3u + v = (0, 4, 3, 3, 1)\,.$$

Verify that for $\overline{x} = (2,1,1,3,1)^T$ and $y = (0,4,3,3,1)^T$ we have $A\overline{x} = b$, $\overline{x} \geq 0$, $Ay = 0$, $y \geq 0$ and $c^T y = 1 > 0$ so that $\overline{x}$ and y constitute a certificate of unboundedness for the LP.

4: Consider the LP where we maximize $z = c_0 + c^T x$ for $x \in \mathbf{R}^5$ subject to $Ax = b$ and $x \geq 0$, where

$$c_0 = 4 \ , \ c = \big(1, 2, 1, -1, 3\big)^T \ , \ A = \begin{pmatrix} 1 & 2 & -1 & 6 & 1 \\ 2 & 1 & 1 & 0 & -4 \\ 2 & 3 & 1 & 2 & -2 \end{pmatrix} \ , \ b = \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} .$$

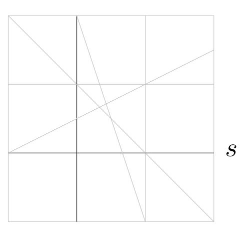
Determine whether the LP is unfeasible, unbounded, or has an optimal solution, and find a certificate.

Solution: First we solve $Ax = b$. We have

$$\big(A\big|b\big) = \left(\begin{array}{ccccc|c} 1 & 2 & -1 & 6 & 1 & 3 \\ 2 & 1 & 1 & 0 & -4 & -3 \\ 2 & 3 & 1 & 2 & -2 & -1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & -1 & 6 & 1 & 3 \\ 0 & 3 & -3 & 12 & 6 & 9 \\ 0 & 1 & -3 & 10 & 4 & 7 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -2 & -3 & -3 \\ 0 & 1 & -1 & 4 & 2 & 3 \\ 0 & 0 & -2 & 6 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3 & -1 & -2 \end{array}\right) .$$

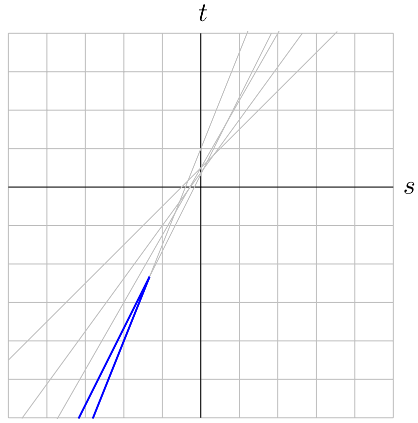
The solution to $Ax = b$ is given by $x = p + su + tv$ where $p = (-1, 1, -2, 0, 0)^T$, $u = (-1, -1, 3, 1, 0)^T$ and $v = (2, -1, 2, 0, 1)^T$. The constraints $x_1 \geq 0, \cdots, x_5 \geq 0$ can be written as $-s + 2t \geq 1$, $-s - t \geq -1$, $3s + t \geq 2$, $s \geq 0$ and $t \geq 0$. We draw a picture to help find the set of points (s, t) which satisfy these constraints. The lines $-s + 2t = 1$, $-s - t = -1$ and $3s + t = 2$ are shown in grey.



We see that there are no points (s, t) which satisfy the constraints, so the LP is unfeasible. A certificate of unfeasibility, is given by a vector y with $A^t y \geq 0$ and $b^T y < 0$. We shall find y with $A^T y \geq 0$ and $b^T y = -1$. First we solve $b^T y = -1$, that is $(3, -3, -1)(y_1, y_2, y_3)^T = -1$. The solution is given by $y_3 = 1 + 3y_1 - 3y_2$, that is $y = p' + su' + tv'$ where $p' = (0, 0, 1)^T$, $u' = (1, 0, 3)^T$ and $v' = (0, 1, -3)^T$. This gives

$$A^T y = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ -1 & 1 & 1 \\ 6 & 0 & 2 \\ 1 & -4 & -2 \end{pmatrix} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \\ -2 \end{pmatrix} + s \begin{pmatrix} 7 \\ 11 \\ 2 \\ 12 \\ -5 \end{pmatrix} + t \begin{pmatrix} -4 \\ -8 \\ -2 \\ -6 \\ 2 \end{pmatrix} .$$

The condition that $A^T y \geq 0$ gives $7s - 4t \geq -2$, $11s - 8t \geq -3$, $2s - 2t \geq -1$, $12s - 6t \geq -2$ and $-5s + 2t \geq 2$. We draw a picture of the set of points (s, t) which satisfy these conditions (shown outlined in blue, bounded by the two lines $12s - 6t = -2$ and $-5s + 2t = 2$).



We see that one solution is given by $(s, t) = (-2, -4)$ corresponding to $y = p' - 2u' - 4v' = (-2, -4, 7)^T$. Verify that $A^T y = (4, 13, 5, 2, 0)^T \geq 0$ and that $b^T y = -1 < 0$ to show that y is a certificate of unfeasibility.

5: Consider the LP where we maximize $z = c_0 + c^T x$ for $x \in \mathbf{R}^5$ subject to $Ax = b$ and $x \geq 0$, where

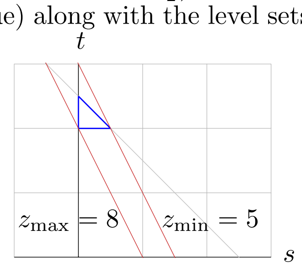
$$c_0 = 2\ ,\ c = \big(-1, 1, 1, -2, 1\big)^T\ ,\ A = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 2 & 0 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 0 \end{pmatrix}\ ,\ b = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

Determine whether the LP is unfeasible, unbounded, or has an optimal solution, and find a certificate.

Solution: First we solve $Ax = b$. We have

$$\big(A\big|b\big) = \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & 1 & 3 \\ 2 & 0 & 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & 1 & 3 \\ 0 & 2 & -1 & 3 & 3 & 5 \\ 0 & 0 & 1 & 1 & 1 & 5 \end{array}\right)$$
$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 1 & 1 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 2 & 5 \\ 0 & 0 & 1 & 1 & 1 & 5 \end{array}\right).$$

The solution to $Ax = b$ is given by $x = p + su + tv$ where $p = (-2, 5, 5, 0, 0)^T$, $u = (0, -2, -1, 1, 0)^T$ and $v = (1, -2, -1, 0, 1)^T$. We need to maximize $z = (c_0 + c^T p) + (c^T u)s + (c^T v)t = 14 - 6s - 3t$ subject to the constraints $x_1 \geq 0, \cdots, x_5 \geq 0$ which we can write as $t \geq 2$, $-2s - 2t \geq -5$, $-s - t \geq -5$, $s \geq 0$ and $t \geq 0$, or equivalently $t \geq 2$, $s + t \leq \frac{5}{2}$, $s \geq 0$ $\big($we can ignore the constraint $t \geq 0$ since we have $t \geq 2$ and we can ignore the constraint $x + y \leq 5$ since we have $x + y \leq \frac{5}{2}\big)$. We draw a picture of the set of points (s, t) which satisfy these constraints (outlined in blue) along with the level sets $z = $min and $z = $max (shown in orange).



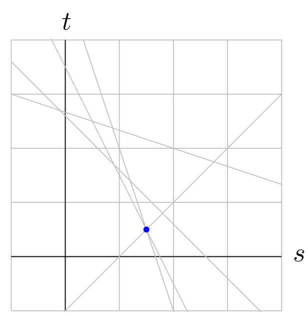
We see that the maximum value of z is $z_{\max} = 8$ and it occurs when $(s, t) = (0, 2)$, that is at the point

$$\overline{x} = p + 0u + 2v = (0, 1, 3, 0, 2)^T.$$

A certificate of optimality for $\overline{x}$ is given by a vector $y \in \mathbf{R}^3$ such that $A^T y \geq c$ and $b^T y = c^T \overline{x}$. First we solve the equation $b^T y = c^T \overline{x}$, that is $3y_1 + y_2 + 2y_3 = 6$. The solution is given by $y_2 = 6 - 3y_1 - 2y_3$, that is by $y = p' + su' + tv'$ where $p' = (0, 6, 0)^T$, $u' = (1, -3, 0)^T$ and $v' = (0, -2, 1)^T$. Then we have

$$A^T y = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \left(\begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 12 \\ 0 \\ 6 \\ 6 \\ -6 \end{pmatrix} + s \begin{pmatrix} -5 \\ 1 \\ -3 \\ -1 \\ 4 \end{pmatrix} + t \begin{pmatrix} -5 \\ -1 \\ -1 \\ -3 \\ 2 \end{pmatrix}.$$

To get $A^T y \geq c$ we need $-5s - 5t \geq -13$, $s - t \geq 1$, $-3s - t \geq -5$, $-s - 3t \geq -8$ and $4s + 2t \geq 7$, or equivalently $s + t \leq \frac{13}{5}$, $s - t \geq 1$, $3s + t \leq 5$, $s + 3t \leq 8$ and $2s + t \geq \frac{7}{2}$. We draw a picture of the set of points (s, t) which satisfy these constraints.



The unique solution is given by $(s, t) = \big(\frac{3}{2}, \frac{1}{2}\big)$, that is $y = p' + \frac{3}{2}u' + \frac{1}{2}v' = \big(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\big)^T$. Verify that $A\overline{x} = b$, $\overline{x} \geq 0$, $A^T y = (2, 1, 1, 3, 1)^T \geq c$ and $b^T y = 6 = c^T \overline{x}$ so that this vector y is indeed a certificate of optimality for the point $\overline{x}$.