CO 250 Intro to Optimization, Solutions to Assignment 1

1: Maximize and minimize $z = c^T x$ for $x \in \mathbf{R}^5$ subject to $Ax = b$ and $x \geq 0$ where

$$c = \big(3, 1, -2, -5, 3\big)^T \ , \ A = \begin{pmatrix} 1 & 2 & 1 & -2 & -3 \\ 1 & 3 & 2 & -2 & -5 \\ 3 & 1 & -1 & -5 & 2 \end{pmatrix} \ , \ b = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} .$$

Solution: We solve $Ax = b$. We have

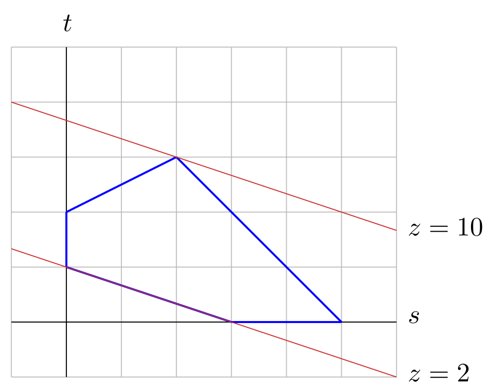
$$\big(A\big|b\big) = \left(\begin{array}{ccccc|c} 1 & 2 & 1 & -2 & -3 & 3 \\ 1 & 3 & 2 & -2 & -5 & 5 \\ 3 & 1 & -1 & -5 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & 1 & -2 & -3 & 3 \\ 0 & 1 & 1 & 0 & -2 & 2 \\ 0 & 5 & 4 & -1 & -11 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & 1 & -1 \\ 0 & 1 & 1 & 0 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 2 & 4 \\ 0 & 1 & 0 & -1 & -3 & -3 \\ 0 & 0 & 1 & 1 & 1 & 5 \end{array}\right)$$

so the solution is $x = p + su + tv$ where $p = (4, -3, 5, 0, 0)^T$, $u = (1, 1, -1, 1, 0)^T$ and $v = (-2, 3, -1, 0, 1)^T$. We must optimize

$$z = c^T x = c \bullet (p + su + tv) = (c \bullet p) + (c \bullet u)s + (c \bullet v)t = -1 + s + 3t$$

subject to the constraints $x_1 \geq 0, x_2 \geq 0, \cdots, x_6 \geq 0$ which we rewrite as $s - 2t \geq -4$, $s + 3t \geq 3$, $-s - t \geq -5$, $s \geq 0$ and $t \geq 0$. We draw a picture of the set of points (s, t) which satisfy these constraints (outlined in blue) along with the level curves $z = \min$ and $z = \max$ (shown in orange).



We see that the minimum value is $z = 2$, which occurs along the line segment from $(s, t) = (0, 1)$ to $(3, 0)$, that is the line segment from $x = (2, 0, 4, 0, 1)^T$ to $(7, 0, 2, 3, 0)^T$, and the maximum value is $z = 10$, which occurs when $(s, t) = (2, 3)$, that is when $x = (4, 8, 0, 2, 3)^T$.

2: Maximize and minimize $z = c^T x$ for $x \in \mathbf{Z}^4$ (this is an IP) subject to $Ax = b$, $x \geq 0$ where

$$c = \big(1, 2, -1, 0\big)^T \ , \ A = \begin{pmatrix} 1 & 2 & 0 & -2 \\ 3 & 2 & 2 & 0 \end{pmatrix} \ , \ b = \begin{pmatrix} -3 \\ 5 \end{pmatrix} .$$

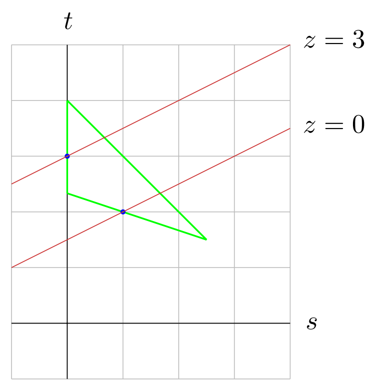
Solution: First we solve $Ax = b$. We have

$$\begin{aligned} (A|b) &= \left(\begin{array}{cccc|c} 1 & 2 & 0 & -2 & -3 \\ 3 & 2 & 2 & 0 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & -2 & -3 \\ 0 & 4 & -2 & -6 & -14 \end{array}\right) \\ &\sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & -2 & -3 \\ 0 & 2 & -1 & -3 & -7 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} & -\frac{7}{2} \end{array}\right) \end{aligned}$$

so the solution is $x = p + su + tv$ where $p = \big(4, -\frac{7}{2}, 0, 0\big)^T$, $u = \big(-1, \frac{1}{2}, 1, 0\big)^T$ and $v = \big(-1, \frac{3}{2}, 0, 1\big)^T$. We must optimize

$$z = c^T x = c \cdot (p + su + tv) = (c \cdot p) + (c \cdot u)s + (c \cdot v)t = -3 - s + 2t$$

subject to the constraints $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$, that is $-s - t \geq 4$, $\frac{1}{2}\, s + \frac{3}{2}\, t \geq \frac{7}{2}$, $s \geq 0$ and $t \geq 0$. The set of all points (s, t) with $s, t \in \mathbf{R}$ which satisfy these constraints is shown below, outlined in green. But we also need to have $x \in \mathbf{Z}$. Since $x_3 = s$ and $x_4 = t$ we need $s, t \in \mathbf{Z}$, and in this case we also have $x_1 \in \mathbf{Z}$ since $x_1 = 4 - s - t$. Since $x_2 = -\frac{7}{2} + \frac{1}{2}s + \frac{3}{2}t$, we see that $x_2 \in \mathbf{Z}$ when $s + t$ is odd. Thus $x \in \mathbf{Z}$ when $s, t \in \mathbf{Z}$ with $s + t$ odd. The only two such pairs (s, t) which satisfy the constraints are the points $(s, t) = (0, 3)$ and $(s, t) = (1, 2)$. Thus the maximum value is $z = 3$, which occurs when $(s, t) = (0, 3)$, that is $x = (1, 1, 0, 3)^T$, and the minimum value is $z = 0$ which occurs when $(s, t) = (1, 2)$, that is $x = (1, 0, 1, 2)^T$.



3: Maximize and minimize $w = x + y + z$ for $(x, y, z)^T \in \mathbf{R}^3$ subject to the non-linear constraints

$$x + 2y - 2z = 1\ ,\ 3x + y^2 + z^2 - 6z \le 4\ ,\ 3x + 5y - z^2 \ge 8\,.$$

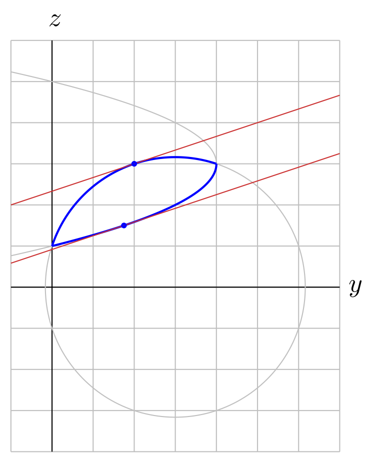
Solution: First we solve the equality $x + 2y - 2z = 1$ to get $x = 1 - 2y + 2z$ with $y, z \in \mathbf{R}$. We put this into the objective function to get

$$w = x + y + z = 1 - 2y + 2z + y + z = 1 - y + 3z$$

and into the inequalities to get

$$\begin{aligned} 3x + y^2 + z^2 - 6z \le 4 &\iff 3 - 6y + 6z + y^2 + z^2 - 6z \le 4 \\ &\iff y^2 - 6y + z^2 \le 1 \iff (y-3)^2 + z^2 \le 10 \\ 3x + 5y - z^2 \ge 8 &\iff 3 - 6y + 6z + 5y - z^2 \ge 8 \\ &\iff z^2 - 6z + y \le -5 \iff (z-3)^2 + y \le 9\,. \end{aligned}$$

We draw a picture of the set of points (y, z) which satisfy these inequalities, outlined in blue $\big($note that $(y-3)^2 + z^2 = 10$ is the equation of the circle of radius $\sqrt{10}$ centred at $(3, 0)$, and $(z-3)^2 + y \le 9$ is the equation of the standard-shaped parabola opening to the left with vertex at $(4, 3)\big)$, along with the level curves $w = \min$ and $w = \max$, shown in orange.



To find the maximum and minimum values of w, we determine the points along the parabola and the circle at which the slope of the tangent line is equal to $\frac{1}{3}$. It is clear that the tangent to the circle at the point $(2, 3)$ will have slope $\frac{1}{3}$ (because the tangent is perpendicular to the radius), and it is clear that the tangent to the parabola at the point $\left(\frac{7}{4}, \frac{3}{2}\right)$ will have slope $\frac{1}{3}$ (since the tangent to the standard parabola $y = x^2$ at the point $\left(\frac{3}{2}, \frac{9}{4}\right)$ has slope 3). When $(y, z) = (2, 3)$ we have $w = 1 - y + 3z = 8$ and when $(y, z) = \left(\frac{7}{4}, \frac{3}{2}\right)$ we have $w = \frac{15}{4}$. Thus the maximum is $w = 8$ which occurs at $(x, y, z) = (3, 2, 3)$ and the minimum is $w = \frac{15}{4}$ which occurs at $(x, y, z) = \left(\frac{1}{2}, \frac{7}{4}, \frac{3}{2}\right)$.

4: Let $A, B, C, u, v \geq 0$.

(a) Suppose we wish to maximize $z = c^T x$ for $x \in \mathbf{R}^n$ subject to the condition that $x \geq 0$ and either $Ax \geq u$ or $Bx \geq v$. Show that this problem can be formulated as an IP.

Solution: We introduce a binary variable t, that is an integer variable $t \in \mathbf{Z}$ with the constraints $t \geq 0$ and $t \leq 1$ so that $t \in \{0, 1\}$. We wish to have $Ax \geq u$ when $t = 0$ and $Bx \geq v$ when $t = 1$, so we include the constraints

$$Ax \geq (1-t)u \ , \ Bx \geq tv \, .$$

Note that when $t = 0$, the constraint $Ax \geq (1-t)u$ becomes $Ax \geq u$ and the constraint $Bx \geq tv$ becomes $Bx \geq 0$, which is satisfied automatically since $B \geq 0$ and $x \geq 0$. Similarly, when $t = 1$ the constraint $Ax \geq (1-t)u$ is satisfied automatically and the constraint $Bx \geq tv$ becomes $Bx \geq v$. Thus the given problem is equivalent to the IP where we maximize $z = c^T x$ for $x \in \mathbf{R}^n$ and $t \in \mathbf{Z}$, subject to the constraints

$$Ax \geq (1-t)u \ , \ By \geq tv \ , \ t \geq 0 \ , \ t \leq 1 \, .$$

These constraints can also be written as

$$\begin{pmatrix} A & u \\ B & -v \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \geq \begin{pmatrix} u \\ v \\ 0 \\ -1 \end{pmatrix} .$$

(b) Suppose that we wish to maximize $z = c^T x$ for $x \in \mathbf{R}^n$ subject to the condition that $x \geq 0$ and at least two of the three matrix inequalities $Ax \geq u$, $Bx \geq v$ and $Cx \geq w$ are satisfied. Show that this problem can be formulated as an IP.

Solution: We introduce binary variables $r, s, t \in \{0, 1\}$. We would like to have $Ax \geq u$ when $r = 1$, $Bx \geq v$ when $s = 1$ and $Cx \geq w$ when $t = 1$, and we would like at least two of the variables r, s, t to be equal to 1, so we include the constraints

$$Ax \geq ru \ , \ Bx \geq sv \ , \ Cx \geq tw \ , \ r + s + t \geq 2 \, .$$

Note that when $r = 0$, the constraint $Ax = ru$ is satisfied automatically since $A \geq 0$ and $x \geq 0$. Similarly, when $s = 0$ the constraint $Bx \geq sv$ is automatically satisfied, and when $t = 0$ the constraint $Cx \geq tw$ is automatically satisfied. Thus the given problem is equivalent to the IP where we maximize $z = c^T x$ for $x \in \mathbf{R}^n$ and $r, s, t \in \mathbf{Z}$ subject to

$$Ax \geq ru \ , \ Bx \geq sv \ , \ Cx \geq tw \ , \ r, s, t \geq 0 \ , \ r, s, t \leq 1 \ , \ r + s + t \geq 2 \, .$$

5: In Conway's game of life, we are given an $n \times n$ grid with cells labeled by pairs (k,l) with $1 \leq k \leq n$ and $1 \leq l \leq n$. Each cell has at most 8 neighbouring cells, where the neighbours of the cell (k,l) are the cells $(k \pm 1, l), (k, l \pm 1), (k \pm 1, l \pm 1)$. Each cell can be either alive or dead. The initial set of living cells is denoted by $L = L_0$. At each stage in the game, the set of living cells changes giving sets $L_0, L_1, L_2, \cdots$. The set L_{n+1} is determined from the set L_n as follows. For each cell (k,l),

if there is at most 1 neighbour of the cell (k,l) which lies in L_n then $(k,l) \notin L_{n+1}$,
if there are exactly 2 neighbours of (k,l) in L_n then $(k,l) \in L_{n+1} \iff (k,l) \in L_n$,
if there are exactly 3 neighbours of (k,l) in L_n then $(k,l) \in L_{n+1}$, and
if there are at least 4 neighbours of (k,l) in L_n then $(k,l) \notin L_{n+1}$.

Suppose that we are given a positive integer n and we wish to find the largest possible size for a set $L = L_0$ with the property that $L_0 = L_1 = L_2 = \cdots$. Show that this problem can be formulated as an IP.

Solution: We first note that if $L_0 = L_1$ then we also have $L_1 = L_2$ because L_2 is obtained from L_1 by the same rules used to obtain L_1 from L_0. Inductively, if $L_0 = L_1$ then we have $L_0 = L_1 = L_2 = \cdots$. Thus the given problem is to find the largest possible size for a set L_0 with the property that $L_0 = L_1$.

Let $S = \{1, 2, \cdots, n\}$ and write $S^2 = \{(i,j) | i \in S, j \in S\}$. We introduce a binary variable $x_{i,j}$ for each pair $(i,j) \in S^2$. The initial set $L_0 \subseteq S^2$ of living cells corresponds to the vector x with entries $x_{i,j}$ with

$$x_{i,j} = \begin{cases} 1 \text{ if } (i,j) \in L \\ 0 \text{ if } (i,j) \notin L. \end{cases}$$

Under this correspondence, the number of elements in L_0 is equal to

$$|L_0| = \sum_{(i,j) \in S^2} x_{i,j} \,.$$

We wish to maximize $|L_0|$ subject to the condition that $L_0 = L_1$. For $(k,l) \in S^2$, let $N(k,l)$ denote the set of neighbours of (k,l). In order to have $L_0 = L_1$, we need the following to hold for each $(k,l) \in S^2$.

If $(k,l) \in L_0$ we require that (k,l) has either 2 or 3 neighbours in L_0, that is $\displaystyle\sum_{(i,j) \in N(k,l)} x_{i,j} \in \{2,3\}$,

If $(k,l) \notin L_0$ then (k,l) cannot have exactly 3 neighbours in L_0, that is $\displaystyle\sum_{(i,j) \in N(k,l)} x_{i,j} \neq 3$.

These constraints are equivalent to

$$\text{If } x_{k,l} = 1 \text{ then } \Big(\sum_{(i,j) \in N(k,l)} x_{i,j} \geq 2 \text{ and } \sum_{(i,j) \in N(k,l)} x_{i,j} \leq 3 \Big)$$
$$\text{If } x_{k,l} = 0 \text{ then } \Big(\sum_{(i,j) \in N(k,l)} x_{i,j} \leq 2 \text{ or } \sum_{(i,j) \in N(k,l)} x_{i,j} \geq 4 \Big).$$

These are equivalent to

$$\sum_{(i,j) \in N(k,l)} x_{i,j} \geq 2\, x_{k,l} \,, \quad \sum_{(i,j) \in N(k,l)} x_{i,j} \leq 8 - 5\, x_{k,l} \,, \quad \Big(\sum_{(i,j) \in N(k,l)} x_{i,j} \leq 2 + 6\, x_{k,l} \text{ or } \sum_{(i,j) \in N(k,l)} x_{i,j} \geq 4 - 4\, x_{k,l} \Big)$$

because for example if $x_{k,l} = 1$ then the first two conditions become $\sum x_{i,j} \geq 2$ and $\sum x_{i,j} \leq 8$ and the last two conditions become $\sum x_{i,j} \leq 8$ and $\sum x_{i,j} \geq 0$ which are both automatically satisfied. Finally, to deal with the disjunction, we introduce another binary variable t, and we use the conditions

$$\sum_{(i,j) \in N(k,l)} x_{i,j} - 2\, x_{k,l} \geq 0 \,, \quad \sum_{(i,j) \in N(k,l)} x_{i,j} + 5\, x_{k,l} \leq 8 \,, \quad \sum_{(i,j) \in N(k,l)} x_{i,j} - 6\, x_{k,l} \leq 2 + 6t \,, \quad \sum_{(i,j) \in N(k,l)} x_{i,j} + 4\, x_{k,l} \geq 4\, t$$

so that for example when $t = 0$ the third condition becomes $\sum x_{i,j} \leq 2 + 6\, x_{k,l}$ while the fourth becomes $\sum x_{i,j} \geq 0$ which is automatically satisfied.