1: (a) Consider the LP where we maximize $z = 3x_1 - x_2 + 2x_3$ for $x_1, x_2, x_3 \in \mathbf{R}$ subject to the constraints $2x_1 + x_2 - x_3 \geq -4, -x_1 + 2x_2 \geq 3$, $x_1 + 3x_2 - x_3 \leq 2$, $-x_1 + 2x_2 - 2x_3 = 1$ and $x_3 \geq 0$. Put the LP into SEF using the variables ${x_1}^+, {x_1}^-, {x_2}^+, {x_2}^-, x_3, s_1, s_2, t$ and then find and simplify the DLP.

(b) Consider the LP where we maximize $z = c^T x$ subject to $Ax \leq b$, $x \geq 0$. Put the LP into SEF then find and simplify the DLP. Show that for feasible points x and y for the LP and the DLP, the complementary slackness conditions are that for all i, either $x_i = 0$ or $(A^T y)_i = c_i$, and for all j, either $y_j = 0$ or $(Ax)_j = b_j$.

2: (a) Consider the LP where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, where

$$c = \big(2, 1, -3, 2, 2, 3\big)^T \ , \ A = \begin{pmatrix} 1 & 3 & 1 & 0 & -4 & 3 \\ 1 & 2 & 1 & 1 & -2 & 4 \\ 2 & 2 & -1 & 1 & -3 & 1 \end{pmatrix} \ , \ b = \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix} .$$

For each of the following points $\overline{x}$, determine whether $\overline{x}$ is an optimal solution to the LP.

$$\overline{x} = (3, 0, 1, 2, 0, 0)^T \ , \ (4, 4, 0, 0, 3, 0)^T \ , \ (1, 0, 0, 3, 0, 4)^T \, .$$

(b) Find an example of an LP, where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, along with a basis B for the LP, such that the basic point $\overline{x}$ for B is an optimal solution to the LP, but the vector $y = {A_B}^{-T} c_B$ is not a certificate of optimality for $\overline{x}$.

3: Let G be the weighted graph with vertex set $V = \{v_1, v_2, v_3, v_4\}$ and edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, where $e_1 = \{v_1, v_2\}$, $e_2 = \{v_1, v_3\}$, $e_3 = \{v_1, v_4\}$, $e_4 = \{v_2, v_3\}$, $e_5 = \{v_2, v_4\}$ and $e_6 = \{v_3, v_4\}$, with weight vector $c = (2, 4, 5, 1, 3, 1)^T$, where $c_i = w(e_i)$.

(a) Let $M = \{S_1, S_2, S_3, S_4\}$ where $S_1 = \{v_1\}$, $S_2 = \{v_1, v_2\}$, $S_3 = \{v_1, v_3\}$ and $S_4 = \{v_1, v_2, v_3\}$. Find $\text{cut}(S)$ for each $S \in M$, and hence find the matrix A with entries

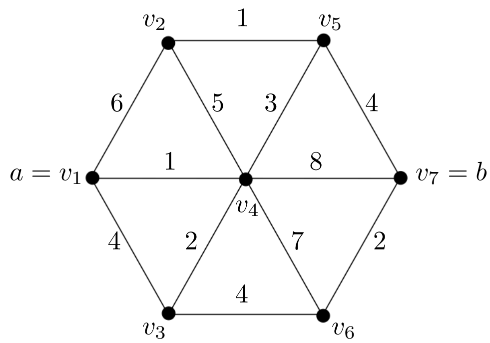
$$A_{S,e} = \begin{cases} 1 \text{ if } e \in \text{cut}(S) \\ 0 \text{ if } e \notin \text{cut}(S) \, . \end{cases}$$

(b) To find the minimum weight path from $a = v_1$ to $b = v_4$, we minimize $z = c^T x$ subject to $Ax \geq \mathbb{1}$, $x \geq 0$. Put this LP into SEF, find and simplify the DLP replacing the dual variable y by $u = -y$, then put the DLP into SEF.

(c) Solve the DLP using Phase II of the Simplex Algorithm, starting with the obvious feasible basis.

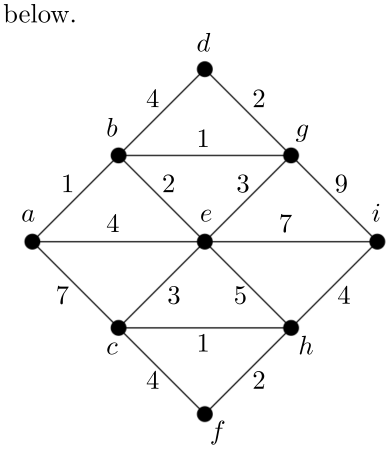
(d) Use your solution from part (c) and the formula for a certificate to obtain an optimal solution to the LP.

4: Let G be the weighted graph with vertex set $V = \{v_1, v_2, \cdots, v_7\}$ and edge set $E = \{e_1, e_2, \cdots, e_{12}\}$, where $e_1 = \{v_1, v_2\}$, $e_2 = \{v_1, v_3\}$, $e_3 = \{v_1, v_4\}$, $e_4 = \{v_2, v_4\}$, $e_5 = \{v_2, v_5\}$, $e_6 = \{v_3, v_4\}$, $e_7 = \{v_3, v_6\}$, $e_8 = \{v_4, v_5\}$, $e_9 = \{v_4, v_6\}$, $e_{10} = \{v_4, v_7\}$, $e_{11} = \{v_5, v_7\}$ and $e_{12} = \{v_6, v_7\}$, with weight vector given by $c = (6, 4, 1, 5, 1, 2, 4, 3, 7, 8, 4, 2)^T$, where $c_i = w(e_i)$ (see the picture below).



Use the Minimum Weight Path Algorithm to find a minimum weight path from $a = v_1$ to $b = v_{12}$ along with an optimal dual solution u. At each step, indicate the vertex set S_k, the cut $\text{cut}(S_k)$, the slack $\text{sl}_k(e)$ for each $e \in \text{cut}(S_k)$, the added edge d_{k+1}, and the value of the entry u_{S_k} of the feasible dual point.

5: Let G be the weighted graph shown below.



(a) Use the Minimum Weight Path Algorithm to find a minimum weight path from a to i along with an optimal dual solution (you can indicate the steps of the algorithm in the form of a picture).

(b) Find an optimal dual solution u with as few nonzero entries u_S as you can.