CO 250 Intro to Optimization, Assignment 4 — Due Tues Mar 12

1: Consider an LP in SEF with constraints $Ax = b$ and $x \geq 0$, where $A \in M_{k\times n}(\mathbf{R})$, $\text{rank}\,(A) = k$ and $b \geq 0$.

(a) For $x \in \mathbf{R}^n$ with $Ax = b$, show that x is a basic point (for some basis B) if and only if $\{A_i \big| x_i \neq 0\}$ is linearly independent, where A_i denotes the i^{th} column of A.

(b) For $x \in \mathbf{R}^n$, show that x is a basic feasible point for the LP if and only if $(x, 0)$ is a basic feasible point for the Auxiliary LP in which we maximize $w(x, s) = -\sum s_i$ subject to $Ax + s = b$ with $x \geq 0$ and $s \geq 0$.

(c) When $A = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 & 0 & -2 & 1 \\ 1 & 2 & 1 & 5 & 4 & 3 & 3 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$, determine which of the following points are basic points: $(1,1,0,0,0,0,0)^T$, $(2,-1,2,0,1,0,0)^T$, $(1,0,1,0,1,0,0)^T$, $(0,0,1,1,0,0,0)^T$, $\left(0,\frac{1}{2},0,0,\frac{1}{2},0,1\right)^T$.

2: Consider the LP in SEF with tableau $\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix}$, where $A \in M_{k\times n}(\mathbf{R})$ and $\text{rank}\,(A) = k$.

(a) Suppose that we apply the Simplex Algorithm to the LP and obtain a basis B whose basic solution $\overline{x}$ maximizes z. Show that $\overline{x}$ together with the vector $y = {A_B}^{-T} c_B$ form a certificate of optimality for the LP.

(b) Suppose that we apply the Simplex Algorithm to the LP, ending with the modified LP with tableau $\begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}$ in canonical form for the basis B, with $\widetilde{c}_l > 0$ and $\widetilde{A}_l \leq 0$ (so that the original LP is unbounded). Show that the basic solution $\overline{x}$ for B together with the vector y given by $y_B = -\widetilde{A}_l$ and $y_N = (e_l)_N$ form a certificate of unboundedness for the original LP.

(c) Suppose that we apply Phase I of the Simplex Algorithm by solving the auxiliary LP in which we maximize $w(x, s) = -\sum s_i$ subject to $Ax + s = b$ with $x \geq 0$ and $s \geq 0$, and we obtain an optimal solution $(x, s) = (\overline{x}, \overline{s})$ with $w(\overline{x}, \overline{s}) = w_{\max} < 0$ (so the original LP is unfeasible). Show that if y is a certificate of optimality for the optimal solution $(\overline{x}, \overline{s})$ for the auxiliary LP, then the same vector y is also a certificate of unfeasibility for the original LP.

3: Consider the LP in SEF with the following tableau

$$\begin{pmatrix} -1 & -2 & 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & 2 & -1 & -1 \\ 0 & 1 & 1 & 3 & -2 & -2 \\ -2 & 0 & -1 & 1 & 1 & 2 \end{pmatrix}.$$

Use the Simplex Algorithm to solve the LP and to find a certificate, as outlined in Problem 2, above.

4: Consider the LP in SEF with the following tableau

$$\begin{pmatrix} -1 & -2 & -2 & 1 & 3 & 1 \\ 1 & 1 & 1 & -2 & -1 & 4 \\ 1 & 1 & 0 & 1 & -2 & 3 \\ -1 & 0 & 1 & -5 & 2 & -3 \end{pmatrix}.$$

Use the Simplex Algorithm to solve the LP and to find a certificate, as outlined in Problem 2.

5: Consider the LP in SEF with the following tableau

$$\begin{pmatrix} -1 & 1 & -1 & -1 & -3 & -2 \\ 1 & -1 & 2 & 1 & 1 & 4 \\ 1 & 1 & -1 & 2 & 4 & 5 \\ -1 & 2 & -1 & 2 & 3 & 4 \end{pmatrix}.$$

Use the Simplex Algorithm to solve the LP and to find a certificate, as outlined in Problem 2.