CO 250 Intro to Optimization, Assignment 2 Due Tues Feb 5

1: (a) Consider the LP where we *minimize* $z(x) = 3 - 2x_1 - x_2 + 2x_3$ subject to the constraints

$$-x_1 - 2x_2 + 3x_3 = 1 \ , \ 3x_1 + x_2 - x_3 = 2$$
$$-2x_1 + 3x_2 - 2x_3 \leq 4 \ , \ -x_1 - x_2 + 2x_3 \geq 3 \ , \ x_1 \leq 0 \, .$$

Convert this to an equivalent LP in SEF for $\widetilde{x} = \left(x_1{}^-, x_2{}^+, x_2{}^-, x_3{}^+, x_3{}^-, s\,, t\,\right)^T \in \mathbf{R}^7$. Express the answer in matrix form (that is in the form where we maximize $\widetilde{z}(\widetilde{x}) = \widetilde{c}_0 + \widetilde{c}^{\,T}$ subject to $\widetilde{A}\,\widetilde{x} = \widetilde{b}$, $\widetilde{x} \geq 0$).

(b) Consider the LP where we maximize $z(x) = 3x_1 - x_2 + 2x_3 + x_4 - 2x_5$ subject to the constraints

$$x_1 + 2x_2 + x_3 + 3x_4 - x_5 = 3 \ , \ 2x_1 + 3x_2 + x_3 + 4x_4 = 5$$
$$3x_1 + 2x_2 + x_3 - x_4 + 6x_5 \leq 4 \ , \ x_1 \geq 0 \ , \ x_3 \geq 0 \, .$$

Solve the equality constraints for x_2 and x_4 in terms of x_1, x_3 and x_5, and then convert this LP to an equivalent LP in SEF for $\widetilde{x} = \left(x_1, x_3, x_5{}^+, x_5{}^-, s\right)^T \in \mathbf{R}^5$. Express the answer in matrix form.

2: An LP in **Standard Inequality Form** (or SIF) is an LP in which we maximize the value of $z(x) = c_0 + c^T x$ for $x \in \mathbf{R}^n$ subject to $Ax \leq b$, where $c_0 \in \mathbf{R}$, $c \in \mathbf{R}^n$ and $A \in M_{k\times n}(\mathbf{R})$.

(a) Show that every LP is equivalent to an LP in SIF by converting the LP where we maximize $z(x) = c_0 + c^T x$ subject to $Ax = u$, $Bx \geq v$, $Cx \leq w$ into an equivalent LP in SIF. Express the answer in matrix form.

(b) Consider the LP in SIF where we maximize $z(x) = c_0 + c^T x$ subject to $Ax \leq b$.

(i) Show that if $y \in \mathbf{R}^k$ with $A^T y = 0$, $y \geq 0$ and $b^T y < 0$, then the LP is unfeasible.

(ii) Show that if $\overline{x} \in \mathbf{R}^n$ with $A\overline{x} \leq b$ and $y \in \mathbf{R}^n$ with $Ay \leq 0$ and $c^T y > 0$ then the LP is unbounded.

(iii) Show that if $\overline{x} \in \mathbf{R}^n$ with $A\overline{x} \leq b$ and $y \in \mathbf{R}^k$ with $A^T y = c$, $y \geq 0$ and $b^T y = c^T \overline{x}$, then $\overline{x}$ is an optimal solution for the LP.

3: Consider the LP where we maximize $z = c_0 + c^T x$ for $x \in \mathbf{R}^5$ subject to the constraints $Ax = b$ and $x \geq 0$, where

$$c_0 = 1 \ , \ c = \left(2, -3, 1, 4, -2\right)^T \ , \ A = \begin{pmatrix} 1 & -1 & 1 & 0 & 1 \\ 1 & 2 & 1 & -3 & -2 \\ 2 & 1 & 1 & -1 & -4 \end{pmatrix} \ , \ b = \begin{pmatrix} 3 \\ -6 \\ -1 \end{pmatrix} .$$

Determine whether the LP is unfeasible, unbounded, or has an optimal solution, and find a certificate.

4: Consider the LP where we maximize $z = c_0 + c^T x$ for $x \in \mathbf{R}^5$ subject to $Ax = b$ and $x \geq 0$, where

$$c_0 = 4 \ , \ c = \left(1, 2, 1, -1, 3\right)^T \ , \ A = \begin{pmatrix} 1 & 2 & -1 & 6 & 1 \\ 2 & 1 & 1 & 0 & -4 \\ 2 & 3 & 1 & 2 & -2 \end{pmatrix} \ , \ b = \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} .$$

Determine whether the LP is unfeasible, unbounded, or has an optimal solution, and find a certificate.

5: Consider the LP where we maximize $z = c_0 + c^T x$ for $x \in \mathbf{R}^5$ subject to $Ax = b$ and $x \geq 0$, where

$$c_0 = 2 \ , \ c = \left(-1, 1, 1, -2, 1\right)^T \ , \ A = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 2 & 0 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 0 \end{pmatrix} \ , \ b = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} .$$

Determine whether the LP is unfeasible, unbounded, or has an optimal solution, and find a certificate.