CO 250 Intro to Optimization, Solutions to Assignment 6

1: Recall that we formalized the maximum weight perfect matching problem using the following LP. Given a weighted graph G, we introduce variables x_e for each edge $e \in E$, and we maximize $z = \sum_{e \in E} c_e x_e$ where $c_e = \text{weight}(e)$ subject to $\sum_{e \in E \text{ s.t. } v \in e} x_e = 1$ for each vertex v and $x_e \geq 0$ for each edge e. Using the Simplex Algorithm to solve this LP, and using our formula for a certificate of optimality, find a maximum weight perfect matching and an optimal dual solution for the weighted graph G with vertex set $V = \{a, b, c, d\}$, edge set $E = \{ab, ac, bc, bd\}$ and weights $c = \left(c_{ab}, c_{ac}, c_{bc}, c_{bd}\right)^T = (5, 4, 6, 3)^T$.

Solution: In matrix form, we must maximize $z = c^T x$ subject to $Ax = \mathbb{1}$, $x \geq 0$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 5 \\ 4 \\ 6 \\ 3 \end{pmatrix}.$$

We put the LP in Canonical form for the basis $\mathcal{B} = \{1, 2, 3, 4\}$ (the only possible basis).

$$\begin{pmatrix} -c^T & 0 \\ A & \mathbb{1} \end{pmatrix} = \begin{pmatrix} -5 & -4 & -6 & -3 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -6 & -3 & 5 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & -5 & -2 & 5 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{15}{2} \\ 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

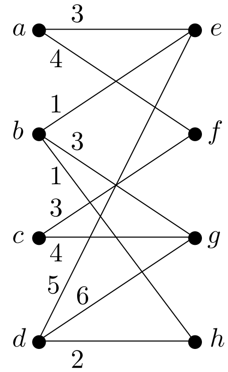
We see that the maximum value for z is $z_{\max} = 7$ and this occurs at $x = (0, 1, 0, 1)^T$. A certificate of optimality (that is a feasible dual solution) is given by $y = A_{\mathcal{B}}^{-T} c_{\mathcal{B}} = A^{-T} c$. We have

$$(A^T | c) = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 5 \\ 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 5 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$$

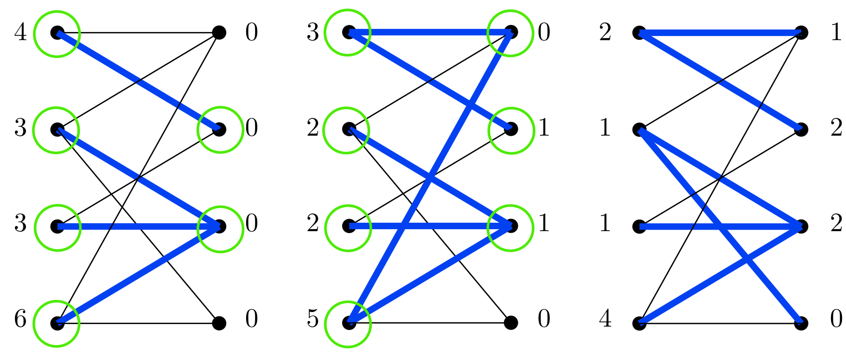
$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & 0 & \frac{7}{2} \\ 0 & 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \end{array}\right)$$

and so $y = \left(\frac{3}{2}, \frac{7}{2}, \frac{5}{2}, -\frac{1}{2}\right)^T$ is an optimal feasible dual solution.

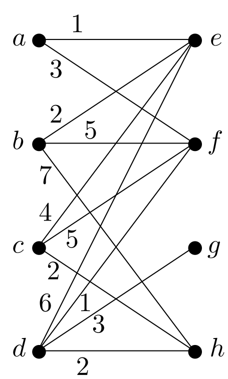
2: (a) Use the Hungarian Algorithm to find a maximum weight perfect matching in the following weighted graph G.



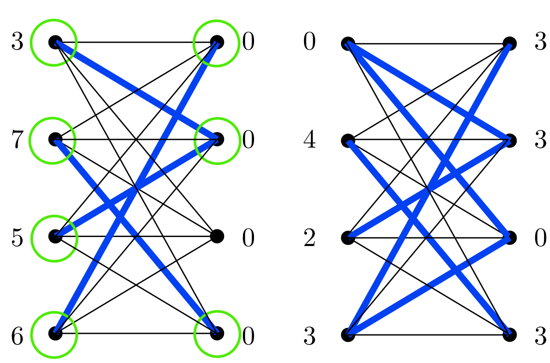
Solution: The steps of the algorithm are summarized in the following pictures. We find that a maximum weight perfect matching is given by $M = \{af, bh, cg, de\}$.



(b) Use the Hungarian Algorithm to find a maximum weight matching in the following weighted graph G.



Solution: First we add edges of weight 0 to obtain the complete bipartite graph $K_{4,4}$, then we apply the algorithm. The steps are summarized in the following pictures. We find that $N = \{ag, bh, cf, de\}$ is a maximum weight perfect matching in $K_{4,4}$, and ag was an added edge so $M = N \setminus \{ag\} = \{bh, cf, de\}$ is a maximum weight matching in G.



3: Consider the IP where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$, where

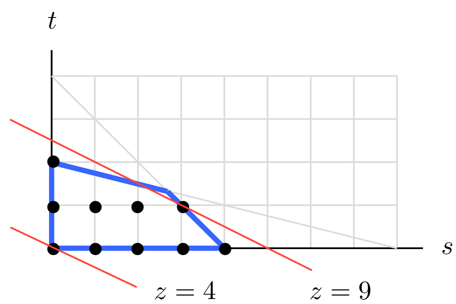
$$A = \begin{pmatrix} 1 & 1 & 2 & 5 \\ 1 & -3 & -2 & 1 \end{pmatrix} , \; b = \begin{pmatrix} 12 \\ -4 \end{pmatrix} , \; c = \big(-2, 5, 4, -1 \big)^T .$$

(a) Find the duality gap for this IP by solving both the IP and its LP relaxation using an accurate sketch of the feasible set.

Solution: We have

$$(A|b) = \left(\begin{array}{cccc|c} 1 & 1 & 2 & 5 & 12 \\ 1 & -3 & -2 & 1 & -4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 2 & 5 & 12 \\ 0 & 4 & 4 & 4 & 16 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 4 & 8 \\ 0 & 1 & 1 & 1 & 4 \end{array}\right)$$

so the solution to $Ax = b$ is given by $x = p + su + tv$ where $p = (8, 4, 0, 0)^T$, $u = (-1, -1, 1, 0)^T$ and $v = (-4, -1, 0, 1)^T$. Also, we have $c = c^T x = c^T(p + su + tv) = (c^T p) + (c^T u)s + (c^T v)t = 4 + s + 2t$. We sketch the feasibility set (outlined in blue) in the st-plane along with some level curves $z = \text{const}$ (in red). From the picture, we see that the maximum value of z for $x \in \mathbf{Z}^4$ is $z = 9$, which occurs at $(s, t) = (3, 1)$, that is at $x = (1, 0, 3, 1)$, and the maximum value of z for $x \in \mathbf{R}^4$ occurs at the point of intersection of the lines $s + 4t = 8$ and $s + t = 4$. The point of intersection is $(s, t) = \big(\frac{8}{3}, \frac{4}{3}\big)$ and then $z = 4 + s + 2t = \frac{28}{3}$. Thus te duality gap is $\frac{28}{3} - 9 = \frac{1}{3}$.



(b) Solve the LP relaxation using the Simplex Algorithm beginning with the feasible basis $\mathcal{B} = \{1, 2\}$, find a cutting-plane and add the corresponding inequality to the constraints, put the new LP into SEF and solve it using the Simplex Algorithm, beginning with a sensibly chosen feasible basis.

Solution: We put the tableau in canonical form for $\mathcal{B} = \{1, 2\}$ then perform iterations of the Simplex Algorithm, indicating the pivots in bold. (We can use the row operations performed in part (a) to save a few steps).

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} 2 & -5 & -4 & 1 & 0 \\ 1 & 0 & 1 & 4 & 8 \\ 0 & 1 & 1 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & -1 & -2 & 4 \\ 1 & 0 & 1 & 4 & 8 \\ 0 & 1 & \mathbf{1} & 1 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 1 & 0 & -1 & 8 \\ 1 & -1 & 0 & \mathbf{3} & 4 \\ 0 & 1 & 1 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & \frac{28}{3} \\ \frac{1}{3} & -\frac{1}{3} & 0 & 1 & \frac{4}{3} \\ -\frac{1}{3} & \frac{4}{3} & 1 & 0 & \frac{8}{3} \end{pmatrix}$$

We see that the maximum value of z for $x \in \mathbf{R}^4$ is $z = \frac{28}{3}$ and this occurs when $x = \big(0, 0, \frac{8}{3}, \frac{4}{3}\big)$. To get a cutting-plane we modify the first equality constraint $\frac{1}{3}x_1 - \frac{1}{3}x_2 + x_4 = \frac{4}{3}$ to get the inequality constraint $\lfloor \frac{1}{3} \rfloor x_1 + \lfloor -\frac{1}{3} \rfloor x_2 + x_4 = \lfloor \frac{4}{3} \rfloor$, that is $-x_2 + x_4 \leq 1$. When we add this constraint to the LP and put the new LP into SEF, we can immediately put the tableau into canonical form for the basis $\widetilde{\mathcal{B}} = \{1, 2, 5\}$ and then we apply the Simplex Algorithm to get

$$\begin{pmatrix} 0 & 0 & -1 & -2 & 0 & 4 \\ 1 & 0 & 1 & 4 & 0 & 8 \\ 0 & 1 & \mathbf{1} & 1 & 0 & 4 \\ 0 & -1 & 0 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 8 \\ 1 & -1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 1 & 1 & 0 & 4 \\ 0 & -1 & 0 & \mathbf{1} & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 9 \\ 1 & 2 & 0 & 0 & -3 & 1 \\ 0 & 2 & 1 & 0 & -1 & 3 \\ 0 & -1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

We see that the new maximum value for z is $z = 9$, and this occurs when $\begin{pmatrix} x \\ s \end{pmatrix} = \big(1, 0, 3, 1, 0\big)^T$, where s is the slack variable. Since $x = (1, 0, 3, 1)^T \in \mathbf{Z}^4$, this is the maximum value of z for the original IP.