CO 250 Intro to Optimization, Solutions to Assignment 5

1: Consider the LP (not in SEF) where we maximize $z = c^T x$ subject to $Ax \le b$ and $x \ge 0$.

(a) Put the LP into SEF then find and simplify the DLP.

Solution: In SEF, we maximize $z = \begin{pmatrix} c^T & 0 \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix}$ subject to $\begin{pmatrix} A & I \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = b$ and $\begin{pmatrix} x \\ s \end{pmatrix} \ge 0$. The DLP is to minimize $w = b^T y$ subject to $\begin{pmatrix} A^T \\ I \end{pmatrix} y \ge \begin{pmatrix} c \\ 0 \end{pmatrix}$. The conditions can be rewritten as $A^T y \ge c$ and $y \ge 0$.

(b) Find optimal solutions $\overline{x}$ and $\overline{y}$ to the LP and the DLP when

$$A = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 4 \end{pmatrix}, \; b = \begin{pmatrix} 2 \\ 10 \\ 5 \end{pmatrix} \text{ and } c = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}.$$

Solution: We solve the LP in SEF using the Simplex Algorithm.

$$\begin{pmatrix} -c^T & 0 & 0 \\ A & I & b \end{pmatrix} = \begin{pmatrix} -2 & 0 & -3 & 0 & 0 & 0 & 0 \\ 2 & 5 & 3 & 1 & 0 & 0 & 2 \\ 3 & 4 & 5 & 0 & 1 & 0 & 10 \\ 1 & 2 & 1 & 0 & 0 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 0 & 5 & 0 & 1 & 0 & 0 & 2 \\ 1 & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 9 \\ 0 & -2 & 1 & -1 & 0 & 1 & 3 \end{pmatrix}$$

We see that the maximum value of z is $z_{\max} = 2$ and it occurs at $\begin{pmatrix} \overline{x} \\ \overline{s} \end{pmatrix} = (1,0,0,0,9,3)^T$ when the basis is $\mathcal{B} = \{1,5,6\}$. An optimal dual solution (that is a certificate for $\overline{x}$) is given by

$$\overline{y} = (A\ I)_{\mathcal{B}}^{-T} \begin{pmatrix} c \\ 0 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

2: Consider the LP where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$ where

$$A = \begin{pmatrix} 1 & 4 & 2 & 15 & 2 & 0 & 7 \\ 0 & 1 & 1 & 6 & 1 & 0 & 3 \\ -1 & 1 & 1 & 6 & 2 & 1 & 6 \\ -5 & -8 & 3 & 2 & 3 & -1 & 1 \end{pmatrix} \ , \ b = \begin{pmatrix} 7 \\ 3 \\ 4 \\ 2 \end{pmatrix} \ \text{and } c = \big(-1, 2, -2, -5, 3, -1, -2 \big)^T .$$

Let $\overline{x} = (1, 0, 3, 0, 0, 2, 0)^T$.

(a) Show that $\overline{x}$ is a basic feasible solution for the basis $\mathcal{B} = \{1, 3, 5, 6\}$ but that the vector $y = {A_{\mathcal{B}}}^{-T} c_{\mathcal{B}}$ is not a certificate of optimality for $\overline{x}$.

Solution: $\overline{x}$ is a feasible solution for the basis $\mathcal{B}$ since $\overline{x} \geq 0$ and $\overline{x}_{\mathcal{N}} = (\overline{x}_2, \overline{x}_4, \overline{x}_7)^T = (0, 0, 0)^T$ and we have

$$A\overline{x} = \begin{pmatrix} 1 & 4 & 2 & 15 & 2 & 0 & 7 \\ 0 & 1 & 1 & 6 & 1 & 0 & 3 \\ -1 & 1 & 1 & 6 & 2 & 1 & 6 \\ -5 & -8 & 3 & 2 & 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 4 \\ 2 \end{pmatrix} = b \,.$$

We have

$$\big(A_{\mathcal{B}}^T \big| c_{\mathcal{B}} \big) = \left(\begin{array}{cccc|c} 1 & 0 & -1 & -5 & -1 \\ 2 & 1 & 1 & 3 & -2 \\ 2 & 1 & 2 & 3 & 3 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & -5 & -1 \\ 0 & 1 & 3 & 13 & 0 \\ 0 & 1 & 4 & 13 & 5 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & -5 & -1 \\ 0 & 1 & 3 & 13 & 0 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -5 & 4 \\ 0 & 1 & 0 & 13 & -15 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 34 \\ 0 & 1 & 0 & 0 & -93 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right)$$

and so $y = A_{\mathcal{B}}^{-T} c_{\mathcal{B}} = (34, -15, 5, 6)^T$, but

$$A^T y = \begin{pmatrix} 1 & 0 & -1 & -5 \\ 4 & 1 & 1 & -8 \\ 2 & 1 & 1 & 3 \\ 15 & 6 & 6 & 2 \\ 2 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 7 & 3 & 6 & 1 \end{pmatrix} \begin{pmatrix} 34 \\ -93 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -2 \\ -6 \\ 3 \\ -1 \\ -5 \end{pmatrix}$$

and so we do not have $A^T y \geq c$.

(b) Show that $\overline{x}$ is an optimal solution and find a certificate of optimality for $\overline{x}$.

Solution: First we put the LP into canonical form for the feasible basis $\mathcal{B} = \{1, 3, 5, 6\}$.

$$\begin{pmatrix} -c^T & 0 \\ A & b \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 & 5 & -3 & 1 & 2 & 0 \\ 1 & 4 & 2 & 15 & 2 & 0 & 7 & 7 \\ 0 & 1 & 1 & 6 & 1 & 0 & 3 & 3 \\ -1 & 1 & 1 & 6 & 2 & 1 & 6 & 4 \\ -5 & -8 & 3 & 2 & 3 & -1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & -6 & 0 & -10 & -5 & 1 & -5 & -7 \\ 1 & 4 & 2 & 15 & 2 & 0 & 7 & 7 \\ 0 & 1 & 1 & 6 & 1 & 0 & 3 & 3 \\ 0 & 5 & 3 & 21 & 4 & 1 & 13 & 11 \\ 0 & 12 & 13 & 77 & 13 & -1 & 36 & 37 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & -6 & 0 & -10 & -5 & 1 & -5 & -7 \\ 1 & 2 & 0 & 3 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 6 & 1 & 0 & 3 & 3 \\ 0 & 2 & 0 & 3 & 1 & 1 & 4 & 2 \\ 0 & -1 & 0 & -1 & 0 & -1 & -3 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 4 & 0 & 5 & 0 & 6 & 15 & 3 \\ 1 & 2 & 0 & 3 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 3 & 0 & -1 & -1 & 1 \\ 0 & 2 & 0 & 3 & 1 & 1 & 4 & 2 \\ 0 & -1 & 0 & -1 & 0 & -1 & -3 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & -2 & 0 & -1 & 0 & 0 & -3 & -9 \\ 1 & 2 & 0 & 3 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 4 & 0 & 0 & 2 & 3 \\ 0 & \mathbf{1} & 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 3 & 2 \end{pmatrix}$$

Then we perform two iterations of the Simplex Algorithm pivoting first at position $(3, 2)$ then at $(3, 7)$.

$$\begin{pmatrix} -c^T & 0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 3 & 2 & 0 & -1 & -9 \\ 1 & 0 & 0 & -1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 4 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 2 & 1 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 5 & 3 & 0 & 0 & -9 \\ 1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 & -2 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -5 & -3 & 1 & 0 & 2 \end{pmatrix}$$

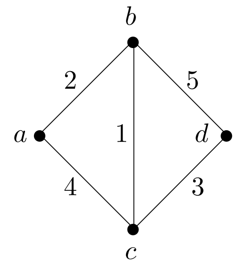
We see that the maximum value of z is $z_{\max} = -9$ and this occurs at $\overline{x} = \big(1, 0, 3, 0, 0, 2, 0\big)^T$ which is the basic solution for the basis $\mathcal{B} = \{1, 3, 6, 7\}$. We have

$$\big(A_{\mathcal{B}}^T | c_{\mathcal{B}}\big) = \left(\begin{array}{cccc|c} 1 & 0 & -1 & -5 & -1 \\ 2 & 1 & 1 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 \\ 7 & 3 & 6 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & -5 & -1 \\ 0 & 1 & 3 & 13 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 3 & 13 & 36 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & -5 & -1 \\ 0 & 1 & 0 & 16 & 3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 4 & -3 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -6 & -2 \\ 0 & 1 & 0 & 16 & 3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 52 \\ 0 & 1 & 0 & 0 & -141 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right)$$

and so $y = \big(52, -141, 8, 9\big)^T$ is a certificate of optimality for $\overline{x}$.

3: Let G be the weighted graph with vertex set $V = \{a, b, c, d\}$, edge set $E = \{ab, ac, bc, bd, cd\}$, and weights given by the vector $c = (2, 4, 1, 5, 3)^T$ (so for example $w(ab) = 2$ and $w(ac) = 4$). Consider the problem of finding the minimum weight path in G from a to d.



(a) Let $M = \big\{S \subseteq V \big| a \in S, d \notin S\big\} = \big\{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\big\}$. Find $\text{cut}(S)$ for each $S \in M$ then find the matrix A with entries

$$A_{S,e} = \begin{cases} 1 \text{ if } e \in \text{cut}(S) \\ 0 \text{ if } e \notin \text{cut}(S). \end{cases}$$

Solution: We have

$$\begin{aligned} \text{cut}\big(\{a\}\big) &= \{ab, ac\} \\ \text{cut}\big(\{a, b\}\big) &= \{ac, bc, bd\} \\ \text{cut}\big(\{a, c\}\big) &= \{ab, bc, bd\} \\ \text{cut}\big(\{a, b, c\}\big) &= \{bd, cd\} \end{aligned}$$

so

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

(b) The minimum weight path problem is to minimize $z = c^T x$ subject to $Ax \geq \mathbb{1}$, $x \geq 0$. Put this LP into SEF, find and simplify the DLP, then put the DLP into SEF.

Solution: In SEF, the LP is to maximize $z = \big(-c^T \ 0\big)\begin{pmatrix} x \\ s \end{pmatrix}$ subject to $\big(A \ -I\big)\begin{pmatrix} x \\ s \end{pmatrix} = \mathbb{1}$ and $\begin{pmatrix} x \\ s \end{pmatrix} \geq 0$. The DLP is to minimize $\mathbb{1}^T y$ subject to $\begin{pmatrix} A^T \\ -I \end{pmatrix} y \geq \begin{pmatrix} -c \\ 0 \end{pmatrix}$ that is subject to $A^T y \geq -c$ and $-y \geq 0$. Letting $u = -y$, we maximize $w = \mathbb{1}^T u$ subject to $A^T u \leq c$ and $u \geq 0$. In SEF the DLP is to maximize $w = \big(\mathbb{1}^T \ 0\big)\begin{pmatrix} u \\ t \end{pmatrix}$ subject to $\big(A^T \ I\big)\begin{pmatrix} u \\ t \end{pmatrix} = c$ and $\begin{pmatrix} u \\ t \end{pmatrix} \geq 0$.

(c) Solve the DLP using the simplex algorithm.

Solution: We have

$$\begin{pmatrix} -\mathbb{1}^T & 0 & 0 \\ A^T & I & c \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 6 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

We see that the maximum value for w is $w_{\max} = 6$ and this occurs at $\begin{pmatrix} \overline{u} \\ \overline{t} \end{pmatrix} = \big(2,1,0,3,0,1,0,1,0\big)^T$ for the basis $\mathcal{B} = \{1,2,4,6,8\}$.

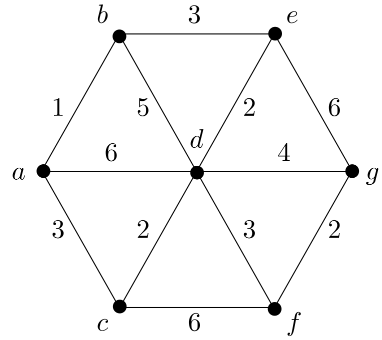
(d) Use your solution from part (c) to find an optimal solution to the LP.

Solution: Since the LP is the dual of the DLP, a certificate $\overline{x}$ for the optimal solution $\overline{u} = (2,1,0,3)^T$ will also be an optimal solution to the LP, so we can take $\overline{x} = \big(A^T\ I\big)_{\mathcal{B}}^{-T} \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}_{\mathcal{B}}$. We have

$$\left((A^T\ I)_{\mathcal{B}}^T \,\middle|\, \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}_{\mathcal{B}} \right) = \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

and so $\overline{x} = \big(1,0,1,0,1\big)^T$ is an optimal solution to the original LP.

4: Consider the problem of finding the minimum weight path from a to g in the weighted graph G shown below.



(a) Let u be the vector with entries u_S for each $S \in M = \{S \subset V | a \in S, g \notin S\}$ whose non-zero entries are

S	$\{a\}$	$\{a,b,c\}$	$\{a,b,c,d\}$	$\{a,b,c,d,e,f\}$
u_S	1	1	2	2

Determine whether u is a feasible dual solution and, if so, whether u is optimal.

Solution: Recall that u is feasible when we have $u_S \geq 0$ for all $S \in M = \{S \subseteq V | a \in S, d \notin S\}$ and also $\mathrm{sl}(e) \geq 0$ for every edge e, where $\mathrm{sl}(e) = c_e - \sum_{S \in M \text{ s.t. } e \in \mathrm{cut}(S)}$. The given vector u is feasible because clearly $u \geq 0$ and the values of $\mathrm{sl}(e)$ are as follows

e	ab	ac	ad	bc	bd	cd	cf	de	df	dg	eg	fg
$\mathrm{sl}(e)$	0	4	2	0	4	1	3	0	1	0	4	0

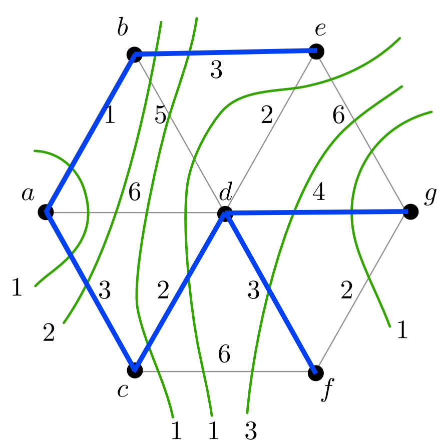
On the other hand, u is not optimal because we can easily find (by inspection) other feasible dual solutions v which give a larger objective value than u. For example, the vector v whose non-zero entries are

S	$\{a,b\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e,f\}$
v_S	3	3	1

is also feasible and $\sum v_S = 7$, which is greater than $\sum u_S = 6$.

(b) Use the algorithm from class to find a minimum weight path in G from a to f along with an optimal dual solution.

Solution: The results of applying the algorithm are shown below. We obtain the minimum-weight path a, c, d, g (with total weight $3 + 2 + 4 = 9$) and the optimal dual solution u given by $u_{\{a\}} = 1$, $u_{\{a,b\}} = 2$, $u_{\{a,b,c\}} = 1$, $u_{\{a,b,c,e\}} = 1$, $u_{\{a,b,c,d,e\}} = 3$, $u_{\{a,b,c,d,e,f\}} = 1$ and $u_S = 0$ for all other sets S.



5: Consider the following problem. Given a graph G with vertex set V and edge set E, we wish to find the vector x, with entries x_e for each $e \in E$, which maximizes the sum $z = \sum_{e\in E} x_e$ subject to the constraints $x_e \geq 0$ for all $e \in E$ and $\sum_{e\in E, v\in e} x_e \leq 2$ for all $v \in V$.

(a) Find A, b and c so that the problem is to maximize $c^T x$ subject to $Ax \leq b$ and $x \geq 0$, then put the LP into SEF.

Solution: In matrix form, we maximize $z = \mathbb{1}^T x$ subject to $Ax \leq \mathbb{2}$ and $x \geq 0$ where $\mathbb{1}$ is the vector whose entries are all 1, $\mathbb{2}$ is the vector whose entries are all 2, and A is the matrix with entries $A_{v,e} = \begin{cases} 1 \text{ if } v \in e \\ 0 \text{ if } v \notin e\,. \end{cases}$

In SEF, we maximize $z = \begin{pmatrix} \mathbb{1}^T & 0 \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix}$ subject to $\begin{pmatrix} A & I \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = \mathbb{2}$ and $\begin{pmatrix} x \\ s \end{pmatrix} \geq 0$.

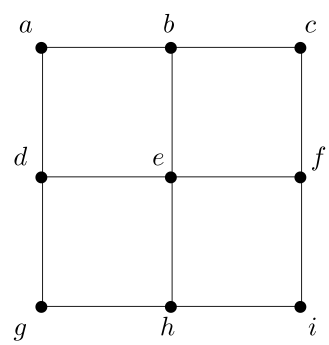
(b) Show that the complementary slackness conditions are that

(1) $\sum_{v\in e} y_v = 1$ for each $e \in E$ with $x_e \neq 0$, and

(2) $y_v = 0$ for each $v \in V$ with $\sum_{e\in E, v\in e} x_e < 2$.

Solution: The complementary slackness conditions are that $\left(\begin{pmatrix} A^T \\ I \end{pmatrix} y \right)_i = \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}_i$ whenever $\begin{pmatrix} x \\ s \end{pmatrix}_i \neq 0$, or equivalently (1) $(A^T y)_e = 1$ whenever $x_e \neq 0$, and (2) $y_v = 0$ whenever $s_v \neq 0$. To simplify (1), note that $(A^T y)_e = \sum_{v\in V} A_{v,e} y_v = \sum_{v\in e} y_v$, and so condition (1) can be rewritten as $\sum_{v\in e} y_v = 1$ whenever $x_e \neq 0$. To simplify (2), note that for feasible x we have $Ax + s = \mathbb{2}$, that is $s = \mathbb{2} - Ax$, and so $s_v = 2 - (Ax)_v = 2 - \sum_{e\in E} A_{v,e} x_e = 2 - \sum_{e\in E \text{ s.t. } v\in e} x_e$. Thus we have $s_v \neq 0 \iff \sum_{e\in E \text{ s.t. } v\in e} x_e \neq 2$, and so condition (2) can be rewritten as $y_v = 0$ whenever $\sum_{e\in E \text{ s.t. } v\in e} x_e < 2$.

(c) For the graph G shown below, find an optimal solution x with $x_e \in \{0, 1\}$ for all $e \in E$, and an optimal dual solution y with $y_v \in \{0, 1\}$ for all $v \in V$.



Solution: First note that, from the SEF form of the given LP, the DLP is to minimize $w = \mathbb{2}^T y$ subject to $\begin{pmatrix} A^T \\ I \end{pmatrix} y \geq \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}$, that is subject to $A^T y \geq \mathbb{1}$ and $y \geq 0$. The vector y has entries y_v for each vertex v, and the DLP is to minimize $w = 2 \sum_{v\in V} y_v$ subject to the conditions that $\sum_{v\in e} y_v \geq 1$ for each edge e and $y_v \geq 0$ for each vertex v. We choose x to be the vector corresponding to the path $a, b, c, f, e, d, g, h, i$, that is

$$x = \left(x_{ab}, x_{ad}, x_{bc}, x_{be}, x_{cf}, x_{de}, x_{dg}, x_{ef}, x_{eh}, x_{fi}, x_{gh}, x_{hi} \right)^T = (1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 1, 1)^T$$

and we choose y to be the vector

$$y = \left(y_a, y_b, y_c, y_d, y_e, y_f, y_g, y_h, y_i \right)^T = (0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)^T\,.$$

It is easy to check that x is feasible for the LP, y is feasible for the DLP, and x and y satisfy the complementary slackness conditions, so they are both optimal solutions. Indeed, $z = \sum_{e\in E} x_e = 8 = 2 \sum_{v\in V} y_v = w$.