CO 250 Intro to Optimization, Solutions to Assignment 4

1: (a) Consider the LP (not in SEF) where we maximize $z = 3x - y - 2$ for $0 \le x, y \in \mathbf{R}$ with $x - 2y \le 1$, $x - y \le 2$, $3x - 2y \le 7$ and $x + y \le 4$. Put this LP into SEF then aply the Simplex Algorithm, beginning with the obvious feasible basis, to solve the LP.

Solution: We give the tableau for the modified LP in SEF, using the variables $\widetilde{x} = (x, y, s_1, \cdots, s_5)^T$, and we apply the Simplex Algorithm. At each stage the entry in the pivot position is shown in bold face.

$$\begin{pmatrix} -\widetilde{c} & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 & 0 & 0 & 0 & -2 \\ \mathbf{1} & -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 & 0 & 2 \\ 3 & -2 & 0 & 0 & 1 & 0 & 7 \\ 1 & 1 & 0 & 0 & 0 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & -5 & 3 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & \mathbf{1} & -1 & 1 & 0 & 0 & 1 \\ 0 & 4 & -3 & 0 & 1 & 0 & 4 \\ 0 & 3 & -1 & 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & -2 & 5 & 0 & 0 & 6 \\ 1 & 0 & -1 & 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & \mathbf{1} & -4 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & -3 & 2 & 0 & 6 \\ 1 & 0 & 0 & -2 & 1 & 0 & 3 \\ 0 & 1 & 0 & -3 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{5} & -2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{4}{5} & \frac{3}{5} & 6 \\ 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & 3 \\ 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{3}{5} & 1 \\ 0 & 0 & 1 & 0 & -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 1 & -\frac{2}{5} & \frac{1}{5} & 0 \end{pmatrix}$$

We see that the maximum value for z is $z_{\max} = 6$ and this value is attained when $\widetilde{x} = \big(3, 1, 0, 0, 0, 0\big)^T$, that is when $(x, y) = (3, 1)$.

(b) Consider the LP where we maximize $z = c^T x$ for $x \in \mathbf{R}^6$ with $Ax = b$ and $x \ge 0$, where

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 3 \\ 2 & 3 & 3 & 1 & 2 & 5 \\ 2 & 5 & 2 & 1 & 3 & 6 \end{pmatrix}, \ b = \begin{pmatrix} 3 \\ 7 \\ 6 \end{pmatrix} \text{ and } c = \big(0, 1, 1, 4, 1, 2\big)^T.$$

Use the Simplex Algorithm, starting with the feasible basis $\mathcal{B} = \{1, 2, 3\}$, to solve the LP.

Solution: We give the tableau for the LP, we perform row operations to put the tableau into canonical form for the basis $\mathcal{B}$, then we perform iterations of the Simplex Algorithm, at each stage showing the entry in the pivot position in bold face.

$$\begin{pmatrix} -\widetilde{c} & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 & -4 & -1 & -2 & 0 \\ 1 & 2 & 1 & 0 & 1 & 3 & 3 \\ 2 & 3 & 3 & 1 & 2 & 5 & 7 \\ 2 & 5 & 2 & 1 & 3 & 6 & 6 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & -1 & -4 & -1 & -2 & 0 \\ 1 & 2 & 1 & 0 & 1 & 3 & 3 \\ 0 & -1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & -2 & -5 & -1 & -1 & -1 \\ 1 & 0 & 3 & 2 & 1 & 1 & 5 \\ 0 & 1 & -1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & -3 & 1 \\ 1 & 0 & 0 & -4 & -2 & \mathbf{4} & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} \frac{3}{4} & 0 & 0 & -4 & -\frac{1}{2} & 0 & \frac{5}{2} \\ \frac{1}{4} & 0 & 0 & -1 & -\frac{1}{4} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & \mathbf{1} & 1 & 0 & 0 \\ \frac{1}{4} & 0 & 1 & 1 & \frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix} \sim \begin{pmatrix} \frac{3}{4} & 4 & 0 & 0 & \frac{7}{2} & 0 & \frac{5}{2} \\ \frac{1}{4} & 1 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ \frac{1}{4} & -1 & 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix}$$

We see that the maximum value for z is $z_{\max} = \frac{5}{2}$ and this value is attained at $\overline{x} = \big(0, 0, \frac{3}{2}, 0, 0, \frac{1}{2}\big)^T$.

2: Consider the LP where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$ where

$$A = \begin{pmatrix} 1 & 2 & -2 & 1 & 0 & 2 \\ 2 & 4 & -2 & 3 & 1 & 3 \\ 1 & 3 & -1 & 2 & 1 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \text{ and } c = \big(1, -1, -2, 3, 2, 3\big)^T.$$

(a) Use Phase I of the Simplex Algorithm to show that the LP is feasible and to find a feasible basis.

Solution: We solve the auxiliary LP where we maximize $w = -\sum s_i = -u^T s$, where $u = (1, 1, 1)^T$, subject to $Ax + s = b$ with $x \geq 0$ and $s \geq 0$. We apply the Simplex Algorithm to the auxiliary LP.

$$\begin{pmatrix} 0 & u & 0 \\ A & I & b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 2 & -2 & 1 & 0 & 2 & 1 & 0 & 0 & 2 \\ 2 & 4 & -2 & 3 & 1 & 3 & 0 & 1 & 0 & 4 \\ 1 & 3 & -1 & 2 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} -4 & -9 & 5 & -6 & -2 & -6 & 0 & 0 & 0 & -7 \\ 1 & 2 & -2 & 1 & 0 & 2 & 1 & 0 & 0 & 2 \\ 2 & 4 & -2 & 3 & 1 & 3 & 0 & 1 & 0 & 4 \\ \mathbf{1} & 3 & -1 & 2 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 3 & 1 & 2 & 2 & -2 & 0 & 0 & 4 & -3 \\ 0 & -1 & -1 & -1 & -1 & \mathbf{1} & 1 & 0 & -1 & 1 \\ 0 & -2 & 0 & -1 & -1 & 1 & 0 & 1 & -2 & 2 \\ 1 & 3 & -1 & 2 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 2 & 0 & 2 & -1 \\ 0 & -1 & -1 & -1 & -1 & 1 & 1 & 0 & -1 & 1 \\ 0 & -1 & \mathbf{1} & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 1 & 4 & 0 & 3 & 2 & 0 & -1 & 0 & 2 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & -1 & 1 & 0 & 1 & -2 & 2 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 1 & 4 & 0 & 3 & 2 & 0 & -1 & 0 & 2 & 0 \end{pmatrix}$$

We see that the maximum value for w is $w_{\max} = 0$, so the original LP is feasible, and that $\mathcal{B} = \{1, 3, 6\}$ is a feasible basis.

(b) Beginning with the feasible basis found in (a), apply Phase II of the Simplex Algorithm to solve the LP.

Solution: From our work in part (a), we know that $(A|b) \sim (\widetilde{A}|\widetilde{b})$ where $\widetilde{b} = (2, 1, 0)^T$ and $\widetilde{A}$ is the lower-left 3×6 matrix in the final tableau above. We then rearrange the rows of $\widetilde{A}$ to get

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} -c^T & c_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 2 & -3 & -2 & -3 & 0 \\ 1 & 4 & 0 & 3 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 & -1 & 1 & 2 \end{pmatrix}$$

We perform the row operation $R_0 \mapsto R_0 + R_1 - 2R_2 + 3R_3$ to put this into canonical form for the basis $\mathcal{B}$, then we perform two iterations of the Simplex Algorithm.

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & -3 & -3 & 0 & 4 \\ 1 & 4 & 0 & \mathbf{3} & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 & -1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & 0 & 0 & -1 & 0 & 4 \\ \frac{1}{3} & \frac{4}{3} & 0 & 1 & \mathbf{\frac{2}{3}} & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ \frac{1}{3} & -\frac{2}{3} & 0 & 0 & -\frac{1}{3} & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} \frac{3}{2} & 7 & 0 & \frac{3}{2} & 0 & 0 & 4 \\ \frac{1}{2} & 2 & 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 1 & 2 \end{pmatrix}$$

We see that the maximum value of z is $z_{\max} = 4$ and this value occurs at $\overline{x} = (0, 0, 1, 0, 02)^T$.

3: Consider the LP in SEF with tableau $\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix}$.

(a) Suppose that we appply the Simplex Algorithm to the LP and obtain a basis $\mathcal{B}$ whose basic solution $\overline{x}$ maximizes z. Show that $\overline{x}$ together with the vector $y = {A_\mathcal{B}}^{-T} c_\mathcal{B}$ form a certificate of optimality for the LP.

Solution: When we apply the Simplex Algorithm, we reduce the tableau for the LP to a modified tableau $\begin{pmatrix} -\widetilde{c} & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}$ which is in canonical form for a feasible basis $\mathcal{B}$, and the basic solution $\overline{x}$ is given by $\overline{x}_\mathcal{B} = \widetilde{b}$ and $\overline{x}_\mathcal{N} = 0$. In the case that the algorithm ends with an optimal solution $\overline{x}$, we have $\widetilde{c} \leq 0$. Recall, from class, that $\widetilde{b} = {A_\mathcal{B}}^{-1} b$ and $\widetilde{c} = c - A^T y$, where $y = A_\mathcal{B}^{-T} c_\mathcal{B}$. For y to be a certificate of optimality for $\overline{x}$ for the original LP, we need $A\overline{x} = b$, $\overline{x} \geq 0$, $c^T \overline{x} = y^T b$ and $A^T y \geq c$. We have $A\overline{x} = b$ and $\overline{x} \geq 0$ since $\overline{x}$ is a feasible point, and we have

$$c^T \overline{x} = {c_\mathcal{B}}^T \overline{x}_\mathcal{B} + {c_\mathcal{N}}^T \overline{x}_\mathcal{N} = {c_\mathcal{B}}^T \overline{x}_\mathcal{B} = {c_\mathcal{B}}^T \widetilde{b} = {c_\mathcal{B}}^T {A_\mathcal{B}}^{-1} b = \left({A_\mathcal{B}}^{-T} c_\mathcal{B}\right)^T b = y^T b$$

and we have $c - A^T y = \widetilde{c} \leq 0$ so that $A^T y \geq c$.

(b) Suppose that we apply the Simplex Algorithm to the LP, ending with the modified LP with tableau $\begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}$ in canonical form for the basis $\mathcal{B}$, with $\widetilde{c}_k > 0$ and $\widetilde{A}_k \leq 0$ (so that the original LP is unbounded). Show that the basic solution $\overline{x}$ for $\mathcal{B}$ together with the vector y given by $y_\mathcal{B} = -\widetilde{A}_k$ and $y_\mathcal{N} = (e_k)_\mathcal{N}$ form a certificate of unboundedness for the original LP.

Solution: Recall, from class, that $\overline{x}$ and y form a certificate of unboundedness for the modified LP, with $y \geq 0$, $\widetilde{A}\, y = 0$ and $\widetilde{c}^T y = \widetilde{c}_k > 0$. Also recall that $\widetilde{A} = {A_\mathcal{B}}^{-1} A$ and $\widetilde{c} = c - A^T {A_\mathcal{B}}^{-T} c_\mathcal{B}$, or equivalently $A = A_\mathcal{B} \widetilde{A}$ and $c = \widetilde{c} + A^T {A_\mathcal{B}}^{-T} c_\mathcal{B}$. We know that $A\overline{x} = b$ and $\overline{x} \geq 0$ because $\overline{x}$ is feasible, and we know that $y \geq 0$. We also have

$$\begin{aligned} Ay &= A_\mathcal{B} \widetilde{A}\, y = A_\mathcal{B} \cdot 0 = 0 \text{ , and} \\ c^T y &= \left(\widetilde{c} + A^T {A_\mathcal{B}}^{-T} c_\mathcal{B}\right)^T y = \widetilde{c}^T y + {c_\mathcal{B}}^T {A_\mathcal{B}}^{-1} A y = \widetilde{c}^T y + 0 = \widetilde{c}_k > 0 \end{aligned}$$

and so $\overline{x}$ and y also form a certificate of unboundedness for the original LP.

(c) Suppose that we apply Phase I of the Simplex Algorithm by solving the auxiliary LP in which we maximize $w = -\sum s_i$ subject to $Ax + s = b$ with $x \geq 0$ and $s \geq 0$, and we obtain an optimal solution $\begin{pmatrix} \overline{x} \\ \overline{s} \end{pmatrix}$ with $w = w_{\max} < 0$ (so the original LP is unfeasible). Show that if y is a certificate of optimality for the optimal solution for the auxiliary LP, then the same vector y is also a certificate of unfeasibility for the original LP.

Solution: Let $u = (1, \cdots, 1)^T$ so that $w = -\sum s_i = -u^T s$ and the auxiliary LP has objective vector $(0, -u^T)$. A certificate of optimality y for the optimal solution $\begin{pmatrix} \overline{x} \\ \overline{s} \end{pmatrix}$ for this auxiliary LP satisfies the conditions $(0, -u)^T \begin{pmatrix} \overline{x} \\ \overline{s} \end{pmatrix} = y^T b$ and $y^T (A, I) \geq (0, -u^T)$. The first condition gives $y^T b = -u^T \overline{s} = -\sum \overline{s}_i = w_{\max}$ so we have $y^T b < 0$, and the second condition gives $y^T A \geq 0$ (and also $y^T \geq -u^T$). Since $y^T b < 0$ and $y^T A \geq 0$, the vector y is a certificate of unboundedness for the original LP.

4: Consider the LP where we maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$ where

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 & 1 & 4 \\ 1 & 1 & 2 & 0 & 1 & 3 \\ 2 & 3 & 4 & 2 & 1 & 5 \end{pmatrix}, \; b = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \text{ and } c = \big(2, 1, 3, 0, 1, 4\big)^T.$$

(a) Use Phase I of the Simplex Algorithm, to show that the LP is unfeasible.

Solution: Let $u = (1, 1, 1)^T$. We begin with the tableau $\begin{pmatrix} 0 & u & 0 \\ A & I & b \end{pmatrix}$ for the auxiliary LP, where we maximize $w = -u^T s$ subject to $Ax + s = b$, $x \geq 0$ and $s \geq 0$, we perform the row operation $R_0 \to R_0 - R_1 - R_2 - R_3$ to put the tableau in canonical form for the basis $\{7, 8, 9\}$, and then we perform several iterations of the Simplex Algorithm.

$$\begin{pmatrix} 0 & u & 0 \\ A & I & b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 & 1 & 4 & 1 & 0 & 0 & 3 \\ 1 & 1 & 2 & 0 & 1 & 3 & 0 & 1 & 0 & 1 \\ 2 & 3 & 4 & 2 & 1 & 5 & 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} -4 & -6 & -9 & -3 & -3 & -12 & 0 & 0 & 0 & -6 \\ 1 & 2 & 3 & 1 & 1 & 4 & 1 & 0 & 0 & 3 \\ 1 & 1 & 2 & 0 & 1 & 3 & 0 & 1 & 0 & 1 \\ 2 & 3 & 4 & 2 & 1 & 5 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & -2 & -1 & -3 & 1 & 0 & 0 & 4 & 0 & -2 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 1 & 2 & 0 & 1 & 3 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & -1 & -1 & 0 & -2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} & 0 & -3 & \frac{3}{2} & \frac{3}{2} & 0 & \frac{9}{2} & 0 & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{3}{2} & 0 & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{3}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 2 & -1 & -1 & 0 & -2 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & \frac{3}{2} & -\frac{3}{2} \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 0 & 1 & -1 & 1 & 2 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 2 & -1 & -1 & 0 & -2 & 1 & 0 \end{pmatrix}$$

We see that the maximum value of w is $w_{\max} = -\frac{3}{2}$, and this occurs at $\big(\overline{x}^T, \overline{s}^T\big) = \big(0, 0, \frac{1}{2}, 0, 0, 0, \frac{3}{2}, 0, 0\big)^T$. Since $w_{\max} < 0$, it follows that the original LP is not feasible.

(b) Use the results of problem 3 to obtain a certificate of unfeasibility for the LP.

Solution: Let $\big(\overline{x}^T, \overline{s}^T\big)$ be the optimal solution for the auxiliary LP found above. Note that it is the basic point for the basis $\mathcal{B} = \{2, 3, 7\}$. By problem 3(a), a certificate of optimality y, for $\big(\overline{x}^T, \overline{s}^T\big)$ for the auxiliary LP, is given by

$$y = (A \; I)_{\mathcal{B}}^{-T} \begin{pmatrix} 0 \\ \mu \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix}^{-T} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

We calculate y as follows.

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 3 & 2 & 4 & 0 \\ ]1 & 0 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 3 & 2 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 2 & 4 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

We find that $y = \big(-1, \frac{1}{2}, \frac{1}{2}\big)^T$. By problem 3(c), this same vector y also serves as a certificate of unfeasibility for the original LP.

5: (a) Consider the LP where we maximize $z = 2x_1 - x_2 + 3x_3$ for $x_1, x_2, x_3 \in \mathbf{R}$ subject to the constraints $x_1 + 3x_2 - x_3 \geq -1$, $2x_1 + x_2 - 4x_3 \leq 3$, $x_1 + 2x_2 + x_3 = 1$ and $x_3 \geq 0$. Put the LP into SEF using the variables ${x_1}^+, {x_1}^-, {x_2}^+, {x_2}^-, x_3, s_1, s_2$, then find and simplify the dual LP.

Solution: The modified LP in SEF is to maximize $z = \widetilde{c}^T \widetilde{x}$ subject to $\widetilde{A}\widetilde{x} = \widetilde{b}$ with $\widetilde{x} \geq 0$, where

$$\widetilde{x} = \left({x_1}^+, {x_1}^-, {x_2}^+, {x_2}^-, x_3, s_1, s_2\right)^T \ , \ \widetilde{c} = \left(2, -2, -1, 1, 3, 0, 0\right)^T \ ,$$

$$\widetilde{A} = \begin{pmatrix} 1 & -1 & 3 & -3 & -1 & -1 & 0 \\ 2 & -2 & 1 & -1 & -4 & 0 & 1 \\ 1 & -1 & 2 & -2 & 1 & 0 & 0 \end{pmatrix} \ , \text{ and } \widetilde{b} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} .$$

The DLP is to minimize $w = \widetilde{b}^T y$ subject to $\widetilde{A}^T y \geq \widetilde{c}$, that is we minimize $w = -y_1 + 3y_2 + y_3$ subject to the constraints $y_1 + 2y_2 + y_3 = 2$, $3y_1 + y_2 + 2y_3 = -1$, $-y_1 - 4y_2 + y_3 \geq 3$, $y_1 \leq 0$ and $y_2 \geq 0$.

(b) Consider the LP where we maximize $z = c^T x$ for $x \in \mathbf{R}^2$ subject to $Ax \leq b$ where

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 2 \\ 2 & -3 \\ -3 & -1 \\ -1 & -1 \end{pmatrix} \ , \ b = \begin{pmatrix} 1 \\ 5 \\ 7 \\ 8 \\ 4 \end{pmatrix} \text{ and } c = \begin{pmatrix} -1 \\ -3 \end{pmatrix} .$$

Put the LP into SEF, find and simplify the dual LP, then put the dual LP into SEF.

Solution: The modified LP in SEF is to maximize $z = \widetilde{c}^T x$ subject to $\widetilde{A}\widetilde{x} = \widetilde{b}$ with $\widetilde{x} \geq 0$ where

$$\widetilde{x} = \left({x_1}^+, {x_1}^-, {x_2}^+, {x_2}^-, s_1, s_2, s_3, s_4, s_5\right)^T \ , \ \widetilde{c} = \left(-1, 1, -3, 3, 0, 0, 0, 0, 0\right)^T \ ,$$

$$\widetilde{A} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 2 & -2 & 0 & 1 & 0 & 0 & 0 \\ 2 & -2 & -3 & 3 & 0 & 0 & 1 & 0 & 0 \\ -3 & 3 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \widetilde{b} = \begin{pmatrix} 1 \\ 5 \\ 7 \\ 8 \\ 4 \end{pmatrix} .$$

The dual LP (or DLP) is to minimize $w = \widetilde{b}^T y$ subject to $\widetilde{A}^T y \geq \widetilde{c}$, or equivalently, to minimize

$$w = (1, 5, 7, 8, 4)\, y$$

subject to

$$\begin{pmatrix} 1 & -1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 & -1 \end{pmatrix} y = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \ , \ y \geq 0 .$$

In SEF, the DLP is to maximize

$$-w = (-1, -5, -7, -8, -4)\, y$$

subject to

$$\begin{pmatrix} 1 & -1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 & -1 \end{pmatrix} y = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \ , \ y \geq 0 .$$