CO 250 Intro to Optimization, Solutions to Assignment 3

1: Consider an LP with constraints $Ax = b$, $x \geq 0$ where

$$A = \begin{pmatrix} 1 & 0 & 1 & -3 & 2 \\ 2 & 1 & 1 & -2 & -1 \\ 1 & 1 & -1 & 2 & -6 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}.$$

Find the first ordered triple in the list

$$(1,2,3),(1,2,4),(1,2,5),(1,3,4),(1,3,5),(1,4,5),(2,3,4),(2,3,5),(2,4,5),(3,4,5)$$

which is a feasible basis for the LP.

Solution: We begin by row-reducing the augmented matrix $(A|b)$ in the standard way, putting the pivots in position2 1, 2 and 3.

$$\big(A\big|b\big) = \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -3 & 2 & 1 \\ 2 & 1 & 1 & -2 & -1 & 3 \\ 1 & 1 & -1 & 2 & -6 & -3 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -3 & 2 & 1 \\ 0 & 1 & -1 & 4 & -5 & 1 \\ 0 & 1 & -2 & 5 & -8 & -4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -3 & 2 & 1 \\ 0 & 1 & -1 & 4 & -5 & 1 \\ 0 & 0 & 1 & -1 & 3 & 5 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & -1 & -4 \\ 0 & 1 & 0 & 3 & -2 & 6 \\ 0 & 0 & 1 & -1 & 3 & 5 \end{array}\right).$$

We see that the basis solution for the basis $\mathcal{B} = \{1,2,3\}$ is $\overline{x} = (-4,6,5,0,0)$ which is not feasible. To find the basic solution for the second basis $\mathcal{B}' = \{1,2,4\}$ we pivot at position $(3,4)$, performing the row-operations $R_1 \mapsto R_1 - 2R_3$, $R_2 \mapsto R_2 + 3R_3$ and $R_3 \mapsto -R_3$ to get

$$\big(A\big|b\big) \sim \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -7 & -14 \\ 0 & 1 & 3 & 0 & 7 & 21 \\ 0 & 0 & -1 & 1 & -3 & -5 \end{array}\right).$$

The corresponding basic solution is $\overline{x}' = (-14,21,0,-5,0)^T$, which is again unfeasible. To find the basic solution for the third basis $\mathcal{B}'' = \{1,2,5\}$ we then pivot at position $(3,5)$, performing the row operations $R_1 \mapsto R_1 - \frac{7}{3}R_3$, $R_2 \mapsto R_2 + \frac{7}{3}R_3$ and $R_3 \mapsto -\frac{1}{3}R_3$, to get

$$\big(A\big|b\big) \sim \left(\begin{array}{ccccc|c} 1 & 0 & \frac{1}{3} & -\frac{7}{3} & 0 & -\frac{7}{3} \\ 0 & 1 & \frac{2}{3} & \frac{7}{3} & 0 & \frac{28}{3} \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} & 1 & \frac{5}{3} \end{array}\right).$$

The corresponding basic point $\overline{x}'' = \big(-\frac{7}{3}, \frac{28}{3}, 0, 0, \frac{5}{3}\big)^T$ is again unfeasible. To find the basic point for the fourth basis $\mathcal{B}''' = \{1,3,4\}$, we go back to the reduced matrix that we used to find the basic point for $\mathcal{B}' = \{1,2,4\}$ and we pivot at position $(2,3)$ to get

$$\big(A\big|b\big) \sim \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -7 & -14 \\ 0 & 1 & 3 & 0 & 7 & 21 \\ 0 & 0 & -1 & 1 & -3 & -5 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & \frac{2}{3} & 0 & 0 & -\frac{7}{3} & 0 \\ 0 & \frac{1}{3} & 1 & 0 & \frac{7}{3} & 7 \\ 0 & \frac{1}{3} & 0 & 1 & -\frac{2}{3} & 2 \end{array}\right).$$

This time the corresponding basic point is $\overline{x}''' = (0,0,7,2,0)^T$, which is feasible. Thus the first feasible basis is $\mathcal{B}''' = \{1,3,4\}$.

We also remark that it is possible to solve this problem graphically.

2: Consider the LP in which we maximize $z = c_0 + c^T x$ subject to $Ax = b$ and $x \geq 0$ where

$$c_0 = 4\ ,\ c^T = \big(1, 1, 2, 1, -1, 3\big)\ ,\ A = \begin{pmatrix} 1 & 2 & 1 & 3 & -1 & 5 \\ 2 & 5 & 3 & 4 & -1 & 7 \\ 1 & 3 & 1 & 2 & 0 & 3 \end{pmatrix}\ ,\ b = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

Put this LP into canonical form for the basis $\mathcal{B} = \{3, 4, 6\}$ in the following two ways.

(a) Perform row operations on the tableau $\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix}$ to obtain $\begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}$.

Solution: We have

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} -1 & -1 & -2 & -1 & 1 & -3 & 4 \\ 1 & 2 & 1 & 3 & -1 & 5 & 2 \\ 2 & 5 & 3 & 4 & -1 & 7 & 3 \\ 1 & 3 & 1 & 2 & 0 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 5 & -1 & 7 & 8 \\ 1 & 2 & 1 & 3 & -1 & 5 & 2 \\ 1 & 1 & 0 & 5 & -2 & 8 & 3 \\ 0 & -1 & 0 & 1 & -1 & 2 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 8 & 0 & 0 & 4 & -3 & 3 \\ 1 & 5 & 1 & 0 & 2 & -1 & -1 \\ 0 & -1 & 0 & 1 & -1 & 2 & 2 \\ -1 & -6 & 0 & 0 & -3 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} -\frac{1}{2} & -1 & 0 & 0 & -\frac{1}{2} & 0 & 6 \\ \frac{1}{2} & 2 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 5 & 0 & 1 & 2 & 0 & -1 \\ -\frac{1}{2} & -3 & 0 & 0 & -\frac{3}{2} & 1 & 1 \end{pmatrix}$$

(b) Calculate ${A_\mathcal{B}}^{-1}$ then find $\widetilde{A}$, $\widetilde{b}$, $\widetilde{c}_0$ and $\widetilde{c}$ using the formulas $\widetilde{A} = {A_\mathcal{B}}^{-1} A$, $\widetilde{b} = {A_\mathcal{B}}^{-1} b$, and $\widetilde{c}_0 = c_0 - y^T b$ and $\widetilde{c} = c + A^T y$ where $y = -{A_\mathcal{B}}^{-T} c_\mathcal{B}$.

Solution: First we calculate $A_\mathcal{B}^{-1}$. We have

$$\big(A_\mathcal{B} \big| I\big) = \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 3 & 4 & 7 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 5 & 8 & 3 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 0 & 3 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 2 & 2 & 1 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -1 & -1 & 4 \\ 0 & 0 & 1 & 1 & \frac{1}{2} & -\frac{5}{2} \end{array}\right).$$

Thus we have

$$\widetilde{A} = A_\mathcal{B}^{-1} A = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & 4 \\ 1 & \frac{1}{2} & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 3 & -1 & 5 \\ 2 & 5 & 3 & 4 & -1 & 7 \\ 1 & 3 & 1 & 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 1 & 5 & 0 & 1 & 2 & 0 \\ -\frac{1}{2} & -3 & 0 & 0 & -\frac{3}{2} & 1 \end{pmatrix}$$

$$\widetilde{b} = A_\mathcal{B}^{-1} b = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & 4 \\ 1 & \frac{1}{2} & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$y = -A_\mathcal{B}^{-T} c_\mathcal{B} = -\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & 4 \\ 1 & \frac{1}{2} & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix}$$

$$\widetilde{c}_0 = c_0 - y^T b = 4 + \begin{pmatrix} 0 & \frac{3}{2} & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 6$$

$$\widetilde{c} = c + A^T y = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 1 \\ 3 & 4 & 2 \\ -1 & -1 & 0 \\ 5 & 7 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{3}{5} \\ \frac{5}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

3: Consider the LP in which we maximize $z = c^T x$ subject to $Ax = b$ and$x \geq 0$ where

$$A = \begin{pmatrix} 1 & 0 & 2 & -1 & -3 \\ 2 & 1 & 2 & -3 & -1 \\ 1 & -1 & 3 & -1 & -6 \end{pmatrix} \; , \; b = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \text{ and } c = \big(2,1,3,0,-4\big)^T .$$

Let $\mathcal{B} = \{1,2,3\}$, $\mathcal{B}' = \{1,2,4\}$, $\mathcal{B}'' = \{1,3,4\}$ and $\mathcal{B}''' = \{1,4,5\}$. Find the basic points $\overline{x}$, $\overline{x}'$, $\overline{x}''$ and $\overline{x}'''$ corresponding to these bases, then find the values $z(\overline{x})$, $z(\overline{x}')$, $z(\overline{x}'')$ and $z(\overline{x}''')$, and determine the optimal solution to the given LP.

Solution: We put the tableau for this LP into canonical form for the basis $\mathcal{B} = \{1,2,3\}$.

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} 2 & -1 & -3 & 0 & 4 & 0 \\ 1 & 0 & 2 & -1 & -3 & 1 \\ 2 & 1 & 2 & -3 & -1 & 3 \\ 1 & -1 & 3 & -1 & -6 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & 1 & -2 & -2 & 2 \\ 1 & 0 & 2 & -1 & -3 & 1 \\ 0 & 1 & -2 & -1 & 5 & 1 \\ 0 & 1 & -1 & 0 & 3 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & -1 & -3 & 3 & 3 \\ 1 & 0 & 2 & -1 & -3 & 1 \\ 0 & 1 & -2 & -1 & 5 & 1 \\ 0 & 0 & 1 & 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & -2 & 1 & 5 \\ 1 & 0 & 0 & -3 & 1 & -3 \\ 0 & 1 & 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & -2 & 2 \end{pmatrix} .$$

We see that the basic point for $\mathcal{B}$ is $\overline{x} = \big(-3,5,2,0,0\big)$ and that $z(\overline{x}) = 5$. Next we put the LP into canonical form for the basis $\mathcal{B}' = \{1,2,4\}$ by pivoting at position $(3,4)$ to get

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 2 & 0 & -3 & 9 \\ 1 & 0 & 3 & 0 & -5 & 3 \\ 0 & 1 & -1 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 & -2 & 2 \end{pmatrix} .$$

We see that the basic solution for $\mathcal{B}'$ is the point $\overline{x}\,' = \big(3,3,0,2,0\big)$ and that $z(\overline{x}\,') = 9$. Now we put the LP into canonical form for the basis $\mathcal{B}'' = \{1,3,4\}$ by pivoting at position $(2,3)$. We obtain

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} 0 & 2 & 0 & 0 & 3 & 15 \\ 1 & 3 & 0 & 0 & 4 & 12 \\ 0 & -1 & 1 & 0 & -3 & -3 \\ 0 & 1 & 0 & 1 & 1 & 5 \end{pmatrix} .$$

The basic solution for $\mathcal{B}''$ is the point $\overline{x}\,'' = \big(12,0,-3,5,0\big)$ and $z(\overline{x}\,'') = 15$. Finally, we put the LP into canonical for for $\mathcal{B}''' = \{1,4,5\}$ by pivoting at $(2,5)$ to get

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 12 \\ 1 & \frac{5}{3} & \frac{4}{3} & 0 & 0 & 8 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 1 & 1 \\ 0 & \frac{2}{3} & \frac{1}{3} & 1 & 0 & 4 \end{pmatrix} .$$

We see that the basic solution for $\mathcal{B}'''$ is the point $\overline{x}\,''' = \big(8,0,0,4,1\big)$ and that $z(\overline{x}\,''') = 12$. We also note that $\overline{x}\,''' \geq 0$ and that all the entries on the top row in this final tableau are non-negative, so $\overline{x}\,'''$ is the optimal solution for the LP. (Note that $\overline{x}\,''$ is not feasible).

4: Consider the LP with tableau

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} 0 & -c_2 & -c_3 & 0 & -c_5 & 0 & c_0 \\ 1 & a_{12} & a_{13} & 0 & a_{15} & 0 & b_1 \\ 0 & a_{22} & a_{23} & 1 & a_{25} & 0 & b_2 \\ 0 & a_{32} & a_{33} & 0 & a_{35} & 1 & b_3 \end{pmatrix}.$$

Note that this LP is in canonical form for the basis $\mathcal{B} = \{1, 4, 6\}$. Suppose that $a_{33} \neq 0$ and let $\mathcal{B}' = \{1, 3, 4\}$.
(a) Find the 4×4 matrix E such that when this LP is put into canonical form for the basis $\mathcal{B}'$, it has tableau $E\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix}$.

Solution: We can put this LP into canonical form for the basis $\mathcal{B}' = \{1, 3, 4\}$ by pivoting at position $(3, 3)$. To do this, we perform the row operations $R_0 \mapsto R_0 + \frac{c_3}{a_{33}} R_3$, $R_1 \mapsto R_1 - \frac{a_{13}}{a_{33}} R_3$, $R_2 \mapsto R_2 - \frac{a_{23}}{a_{33}} R_3$ and $R_3 \mapsto \frac{1}{a_{33}} R_3$. Performing these row operations is equivalent to multiplying on the left by the product of the corresponding elementary matrices, so we can take

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{a_{33}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{a_{23}}{a_{33}} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{a_{13}}{a_{33}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \frac{c_3}{a_{33}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \frac{c_3}{a_{33}} \\ 0 & 1 & 0 & -\frac{a_{13}}{a_{33}} \\ 0 & 0 & 1 & -\frac{a_{23}}{a_{33}} \\ 0 & 0 & 0 & \frac{1}{a_{33}} \end{pmatrix}$$

(b) Find necessary and sufficient conditions on a_{ij}, b_i and c_i in order that $\mathcal{B}'$ is feasible.

Solution: When we perform these operations to put the LP into canonical form for the basis $\mathcal{B}'$, we obtain

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} \sim \begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix} \qquad \text{with } \ \widetilde{b} = \begin{pmatrix} b_1 - \frac{a_{13} b_3}{a_{33}} \\ b_2 - \frac{a_{23} b_3}{a_{33}} \\ \frac{b_3}{a_{33}} \end{pmatrix}.$$

The basis $\mathcal{B}'$ is feasible if and only if $\widetilde{b} \geq 0$, that is if and only if $b_1 \geq \dfrac{a_{13} b_3}{a_{33}}$, $b_2 \geq \dfrac{a_{23} b_3}{a_{33}}$ and $\dfrac{b_3}{a_3} \geq 0$.

5: Consider the LP with tableau

$$\begin{pmatrix} -c^T & c_0 \\ A & b \end{pmatrix} = \begin{pmatrix} -1 & 0 & -2 & -5 & 0 & -4 & 6 & 0 & 0 & -2 & 0 & 3 \\ 2 & 0 & 3 & 2 & 0 & 1 & 2 & 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & -3 & 0 & -1 & 0 & 0 & 0 & 4 & -2 & 0 \\ 1 & 0 & -2 & 4 & 0 & 3 & 1 & 0 & 1 & 1 & 1 & 2 \\ -2 & 0 & 1 & -2 & 1 & 0 & -3 & 0 & 0 & 3 & -3 & 0 \end{pmatrix}.$$

Note that, up to a permutation of the rows, the LP is in canonical form for the basis $\mathcal{B} = \{2, 5, 8, 9\}$.
(a) For which choices of $k \notin \mathcal{B}$ and $l \in \mathcal{B}$ is $\mathcal{B}' = \big(\mathcal{B} \cup \{k\}\big) \setminus \{l\}$ a basis for the given LP?

Solution: Let A_j denote the j^{th} column of the matrix A and let $a_{i,j}$ denote the entry in position (i, j). Note that $A_2 = e_2$, $A_5 = e_4$, $A_8 = e_1$ and $A_9 = e_3$ where the e_i are the standard basis vectors for $\mathbf{R}^4$. Let $i_2 = 2$, $i_5 = 4$, $i_8 = 1$ and $i_9 = 3$ so that we have $A_l = e_{i_l}$ for all $l \in \mathcal{B}$. For $l \in \mathcal{B}$ and $k \notin \mathcal{B}$, if we remove A_l from the set $\{A_2, A_5, A_8, A_9\}$ and replace it by the column A_k, then the resulting set of vectors $\big(\{A_2, A_5, A_8, A_9\} \cup \{A_k\}\big) \setminus \{A_l\}$ will be linearly independent if and only if $a_{i_l,k} \neq 0$. In other words, the set $\mathcal{B}' = \big(\mathcal{B} \cup \{k\}\big) \setminus \{l\}$ is a basis for the LP if and only if $a_{i_l,k} \neq 0$. The pairs (i, k) with $k \notin \mathcal{B}$ and $a_{i,k} = 0$ are

$$(i, k) = (2, 1), (4, 6), (2, 7)$$

(these are the positions where we *cannot* pivot to change to a new basis). These correspond to the pairs

$$(k, l) = (1, 2), (6, 5), (7, 2)$$

respectively, where $(i, k) = (i_l, k)$ corresponds to (k, l). There are 28 pairs (k, l), with $k \notin \mathcal{B}$ and $l \in \mathcal{B}$. For all of these pairs except for the 3 pairs listed above, the set $\mathcal{B}' = \big(\mathcal{B} \cup \{k\}\big) \setminus \{l\}$ is a basis for the given LP.

(b) For which of the choices in part (a) is the new basis $\mathcal{B}'$ feasible?

Solution: When we pivot at position (i,k), with $k \notin \mathcal{B}$ and $a_{ik} \neq 0$, to put the LP into canonical form for the basis $\mathcal{B}'$ obtaining the tableau $\begin{pmatrix} -\widetilde{c}^T & \widetilde{c}_0 \\ \widetilde{A} & \widetilde{b} \end{pmatrix}$, the new basis $\mathcal{B}'$ is feasible if and only if $\widetilde{b} \geq 0$. The vector $\widetilde{b}$ is given by $\widetilde{b}_i = \frac{b_1}{a_{ik}}$ and $\widetilde{b}_j = b_j - \frac{a_{jk}b_i}{a_{ik}}$ for $j \neq i$. Thus the new basis $\mathcal{B}'$ is feasible when $\frac{b_i}{a_{ik}} \geq 0$ and $b_j \geq \frac{a_{jk}b_i}{a_{ik}}$ for all $j \neq i$. When $b_i = 0$ both conditions are satisfied, so we obtain a feasible basis when we pivot at any of the positions

$$(i,k) = (2,3),(2,4),(2,6),(2,10),(2,11),(4,1),(4,3),(4,4),(4,7),(4,10),(4,11)\,.$$

When $b_i > 0$, the two conditions are equivalent to the conditions $a_{ik} > 0$ and $\frac{b_i}{a_{ik}} = \min\left\{\frac{b_j}{a_{jk}} \middle| a_{jk} > 0\right\}$. For each $k \notin \mathcal{B}$, the value(s) of i which yield the minimum ratio are as follows

$$\begin{pmatrix} k & 1 & 3 & 4 & 6 & 7 & 10 & 11 \\ i & 1 & 4 & 1,3 & 3 & 1 & 2,4 & 1 \end{pmatrix}$$

and so in addition to the 11 pairs (i,k) already listed above, we can also pivot at positions

$$(i,k) = (1,1),(1,4),(3,4),(3,6),(1,7),(1,11)\,.$$

The corresponding choices for (k,l) are

$$\begin{aligned}(k,l) =\,&(3,2),(4,2),(6,2),(10,2),(11,2),(1,5),(3,5),(4,5),(7,5),(10,5),(11,5),\\ &(1,8),(4,8),(4,9),(6,9),(7,8),(11,8)\,.\end{aligned}$$

(c) For which of the choices in part (b) do we have $z(\overline{x}) = z(\overline{x}')$ (where $\overline{x}$ is the basic point for $\mathcal{B}$ and $\overline{x}'$ is the basic point for $\mathcal{B}'$)?

Solution: We have $z(\overline{x}) = c_0 = 3$, and when we pivot at position (i,k) we have $z(\overline{x}') = \widetilde{c}_0 = 3 + \frac{c_k b_i}{a_{ik}}$, and so $z(\overline{x}) = z(\overline{x}')$ if and only if $c_k b_i = 0$, or equivalently, if and only if $c_k = 0$ or $b_i = 0$, that is $k = 11$ or $i \in \{2,4\}$. The suitable pairs (i,k) from part (b) are

$$(i,k) = (2,3),(2,4),(2,6),(2,10),(2,11),(4,1),(4,3),(4,4),(4,7),(4,10),(4,11),(1,11)$$

and the corresponding pairs (k,l) are

$$(k,l) = (3,2),(4,2),(6,2),(10,2),(11,2),(1,5),(3,5),(4,5),(7,5),(10,5),(11,5),(11,8)\,.$$

(d) For which of the choices in part (b) do we have $z(\overline{x}') > z(\overline{x})$?

Solution: We have $z(\overline{x}) = c_0 = 3$, and when we pivot at position (i,k) we have $z(\overline{x}') = \widetilde{c}_0 = 3 + \frac{c_k b_i}{a_{ik}}$, and so $z(\overline{x}') > z(\overline{z})$ if and only if $\frac{c_k b_i}{a_{ik}} > 0$. To get $\frac{c_k b_i}{a_{ik}} > 0$, we cannot have $b_i = 0$ so we cannot have $i = 2,4$, and the remaining possible pivot positions (i,k) from part (b) are $(i,k) = (1,1),(1,4),(3,4),(3,6),(1,7)$. At each of these positions we have $a_{ik} > 0$ and $b_i > 0$ and so to get $\frac{c_k b_i}{a_{ik}} > 0$ we must have $c_k > 0$. Thus the acceptable pivot positions are $(i,k) = (1,1),(1,4),(3,4),(3,6)$ corresponding to

$$(k,l) = (1,8),(4,8),(4,9),(6,9)\,.$$

(e) Which of the choices in part (d) gives the maximum increase $\Delta z = z(\overline{x}') - z(\overline{z})$.

Solution: We have

(i,k)	$(1,1)$	$(1,4)$	$(3,4)$	$(3,6)$
$\Delta z = \frac{c_k b_i}{a_{ik}}$	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{8}{3}$

so the maximum possible increase is $\Delta z = \frac{8}{3}$, and this occurs when $(i,k) = (3,6)$, that is when $(k,l) = (6,9)$.