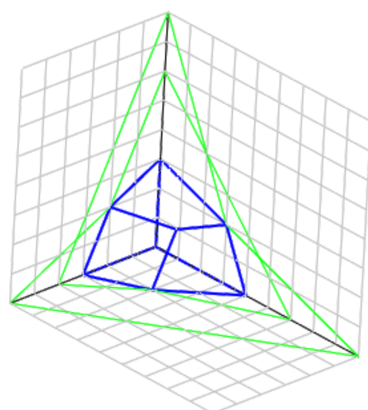
CO 250 Intro to Optimization, Solutions to Assignment 2

1: Consider the LP where we maximize $z = c_0 + c^T x$ for $x \in \mathbf{R}^3$ subject to $Ax \leq b$ and $x \geq 0$ where

$$c_0 = 1 \ , \ c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \ , \ A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 6 \end{pmatrix} \text{ and } b = \begin{pmatrix} 6 \\ 8 \\ 18 \end{pmatrix} .$$

(a) Draw an accurate picture of the feasible set in $\mathbf{R}^3$.

Solution: The feasible set is the intersection of the half-spaces $2x_1 + x_2 + x_3 \leq 6$, $2x_1 + 2x_2 + x_3 \leq 8$ and $3x_1 + 2x_2 + 6x_3 \leq 18$ with the positive octant (given by $x_1, x_2, x_3 \geq 3$). In green, we show the portion of each of the three planes $2x_1 + x_2 + x_3 = 6$, $2x_1 + 2x_2 + x_3 = 8$ and $3x_1 + 2x_2 + 6x_3 = 18$ which lies in the positive octant, then we show the edges of the feasible polyhedron in blue. To make the picture accurate, we should calculate the coordinates of the nearest vertex. The calculation is done below in part (b).



(b) Find the exact coordinates of all the vertices of the feasible set.

Solution: Seven of the eight vertices can be seen immediately from the picture in part (a), namely

$$(0,0,0) \ , \ (3,0,0) \ , \ (2,2,0) \ , \ (0,4,0) \ , \ (0,3,2) \ , \ (0,0,3) \ , \ (2,0,2) .$$

The last vertex is the intersection of the three planes, which we now calculate.

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 6 \\ 2 & 2 & 1 & 8 \\ 3 & 2 & 6 & 18 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 3 \\ 2 & 2 & 1 & 8 \\ 3 & 2 & 6 & 18 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 3 \\ 0 & 1 & 0 & 2 \\ 0 & \frac{1}{2} & \frac{9}{2} & 9 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & \frac{9}{2} & 8 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{16}{9} \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{10}{9} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{16}{9} \end{array}\right)$$

Thus the final vertex is at $\left(\frac{10}{9}, 2, \frac{16}{9}\right)$.

(c) Solve the LP.

Solution: Since $z = c_0 + c^T x$ with $c_0 = 1$ and $c^T = (1,1,1)$, that is $z(x_1, x_2, x_3) = 1 + x_1 + x_2 + x_3$, the value of z at each of the vertices is as follows: $z(0,0,0) = 1$, $z(3,0,0) = 4$, $z(2,2,0) = 5$, $z(0,4,0) = 5$, $z(0,3,2) = 6$, $z(0,0,3) = 4$, $z(2,0,2) = 5$ and $z\left(\frac{10}{9}, 2, \frac{16}{9}\right) = \frac{53}{9} < 6$, and so the maximum value of z is $z(0,3,2) = 6$.

2: (a) Consider the LP where we *minimize* $z = x_1 - 2x_2 - x_3 - 5$ subject to the constraints $x_1 + 2x_2 - x_3 = 3$, $2x_1 - 5x_2 + 3x_3 = 1$, $x_1 + 3x_2 + 2x_3 \leq 4$, $-x_1 - x_2 + 3x_3 \geq -1$ and $x_2 - 2x_3 \geq 1$. Convert this to an equivalent LP in SEF.

Solution: We write $x_i = x_i^+ - x_i^-$ for $i = 1, 2, 3$ with $x_i + \geq 0$ and $x_i^- \geq 0$. We also introduce variables $s_1, t_1, t_2 \geq 0$. The given LP is equivalent to the LP where we *maximize*

$$z = 5 - x_1 + 2x_2 + x_3 = 5 - x_1^+ + x_1^- + 2x_2^+ - 2x_2^- + x_3^+ - x_3^-$$

subject to

$$\begin{array}{rrrrrrrrrl} x_1^+ & -x_1^- & 2x_2^+ & -2x_2^- & -x_3^+ & +x_3^- & & & & = 3 \\ 2x_1^+ & -2x_1^- & -5x_2^+ & +5x_2^- & +3x_3^+ & -3x_3^- & & & & = 1 \\ x_1^+ & -x_1^- & +3x_2^+ & -3x_2^- & +2x_3^+ & -2x_3^- & +s_1 & & & = 4 \\ -x_1^+ & +x_1^- & -x_2^+ & +x_2^- & +3x_3^+ & -3x_3^- & & -t_1 & & = -1 \\ & & x_2^+ & -x_2^- & -2x_3^+ & +2x_3^- & & & -t_2 & = 1 \end{array}$$

(b) Consider the LP where we maximize $z = c_0 + c^T x$ for $x \in \mathbf{R}^5$ subject to the constraints $Ax = b$ and $A'x \geq b'$ where $c_0 = 6$ and

$$c = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 1 \end{pmatrix}, \ A = \begin{pmatrix} 1 & 2 & 1 & 3 & 2 \\ 2 & 4 & 1 & 2 & 3 \end{pmatrix}, \ A' = \begin{pmatrix} 1 & 2 & 0 & 1 & 4 \\ 1 & 3 & 1 & 2 & 1 \end{pmatrix}, \ b = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \ b' = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

By first solving the equation $Ax = b$, convert this to an equivalent LP where we must maximize $z = \widetilde{c}_0 + \widetilde{c}^T \widetilde{x}$ for $\widetilde{x} \in \mathbf{R}^8$ subject to $\widetilde{A}\,\widetilde{x} = \widetilde{b}$, $\widetilde{x} \geq 0$, where $\widetilde{c} \in \mathbf{R}^8$ and $\widetilde{b} \in \mathbf{R}^2$.

Solution: First we solve $Ax = b$. We have

$$(A|b) = \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 2 & 1 \\ 2 & 4 & 1 & 2 & 3 & 5 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 4 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 1 & 4 \\ 0 & 0 & 1 & 4 & 1 & -3 \end{array}\right).$$

The solution is

$$x = p + t_1 u_2 + t_2 u_2 + t_3 u_3$$

where $p = (4, 0, -3, 0, 0)^T$, $u_1 = (-2, 1, 0, 0, 0)^T$, $u_2 = (1, 0, -4, 1, 0)^T$ and $u_3 = (-1, 0, -1, 0, 1)^T$. We have

$$\begin{aligned} z &= c_0 + c^T x = c_0 + c^T\big(p + t_1 u_1 + t_2 u_2 + t_3 u_3\big) = (c_0 + c^T p) + (c^T u_1)t_1 + (c^T u_2)t_2 + (c^T u_3)t_3 \\ &= (6 - 2) + t_1 - 3t_2 - 2t_3 = 4 + t_1^+ - t_1^- - 3t_2^+ + 3t_2^- - 2t_3^+ + 2t_3^- \end{aligned}$$

where we have written $t_i = t_i^+ - t_i^-$ for $i = 1, 2, 3$ with $t_i^+, t_i^- \geq 0$. Also, we have

$$\begin{aligned} A'x \geq b' &\iff A'\big(p + t_1 u_2 + t_2 u_2 + t_3 u_3\big) \geq b' \iff (A'u_1)t_1 + (A'u_2)t_2 + (A'u_3)t_3 \geq (b' - A'p) \\ &\iff \begin{pmatrix} 0 \\ 1 \end{pmatrix} t_1 + \begin{pmatrix} 2 \\ -1 \end{pmatrix} t_2 + \begin{pmatrix} 3 \\ -1 \end{pmatrix} t_3 \geq \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &\iff 2t_2 + 3t_3 \geq -1 \text{ and } t_1 - t_2 - t_3 \geq -2 \\ &\iff 2t_2^+ - 2t_2^- + 3t_3^+ - 3t_3^- - s_1 = -1 \text{ and } t_1^+ - t_1^- - t_2^+ + t_2^- - t_3^+ + t_3^- - s_2 = -2 \end{aligned}$$

where we have introduced two slack variables $s_1, s_2 \geq 0$. Thus the original LP is equivalent to the LP where we maximize $z = \widetilde{c}_0 + \widetilde{c}^T \widetilde{x}$ for $\widetilde{x} = \big(t_1^+, t_1^-, t_2^+, t_2^-, t_3^+, t_3^-, s_1, s_2\big)^T$, subject to $\widetilde{A}\widetilde{x} = \widetilde{b}$ and $\widetilde{x} \geq 0$ where

$$\widetilde{c}_0 = 4\,,\ \widetilde{c} = \big(1, -1, -3, 3, -2, 2, 0, 0\big)^T\,,\ \widetilde{A} = \begin{pmatrix} 0 & 0 & 2 & -2 & 3 & -3 & -1 & 0 \\ 1 & -1 & -1 & 1 & -1 & 1 & 0 & -1 \end{pmatrix}, \text{ and } \widetilde{b} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

3: Consider the following three LPs where we maximize $z = c^T x$ subject to $Ax = b$, $x \geq 0$.

$$\text{LP1} : A = \begin{pmatrix} 1 & 2 & -1 & -4 & 6 \\ 1 & 1 & 0 & -1 & 2 \\ 1 & 1 & -1 & -2 & 5 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \quad , \quad c^T = \big(3, 1, 2, -1, 1\big)$$

$$\text{LP2} : A = \begin{pmatrix} 1 & 1 & -1 & -2 & 3 \\ 0 & 1 & 1 & -1 & -2 \\ 1 & 0 & -1 & 0 & 2 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} \quad , \quad c^T = \big(2, 0, -1, -2, 3\big)$$

$$\text{LP3} : A = \begin{pmatrix} 1 & 1 & -1 & -2 & 3 \\ -1 & 1 & 0 & -3 & -2 \\ -2 & 1 & 1 & -3 & -6 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 2 \\ -5 \\ -9 \end{pmatrix} \quad , \quad c^T = \big(2, 0, 1, 4, -3\big)$$

One of these LPs is infeasible, one is unbounded, and one has an optimal solution. Determine which is which by selecting suitable certificates from amongst the following vectors

$$\overline{x}^T, y^T \in \Big\{(1, 0, 5, 1, 2) \ , \ (0, 3, 0, 2, 1) \ , \ (3, 0, 5, 2, 3) \ , \ (-1, 8, -2) \ , \ \tfrac{1}{4}\big(16, -39, 23\big)\Big\}$$

Solution: For the certificate of unfeasibility we need $y \in \mathbf{R}^3$ with $y^T A \geq 0$ and $y^T b < 0$. For the certificate of unboundedness we need $\overline{x} \in \mathbf{R}^5$ with $a\overline{x} = b$ and $\overline{x} \geq 0$ and $y \in \mathbf{R}^5$ with $Ay = 0$, $y \geq 0$ and $c^T y > 0$. For the certificate of optimality we need $\overline{x} \in \mathbf{R}^5$ with $A\overline{x} = b$ and $\overline{x} \geq 0$ and $y \in \mathbf{R}^3$ with $y^T b = c^T \overline{x}$ and $y^T A \geq c^T$. After some trial and error we find certificates as follows. For LP2 we can use $\overline{x} = (3, 0, 5, 2, 3)^T$ and $y = (1, 0, 5, 1, 2)^T$ as a certificate of unboundedness because $A\overline{x} = b$, $\overline{x} \geq 0$, $Ay = 0$, $y \geq 0$ and $c^T y = 1 > 0$. For LP3 we can use $\overline{x} = (0, 3, 0, 2, 1)^T$ and $y = \frac{1}{4}(16, -39, 23)^T$ as a certificate of optimality since $A\overline{x} = b$, $x \geq 0$, $y^T b = 5 = c^T \overline{x}$ and $y^T A = \frac{1}{4}(11, 0, 7, 16, -12) \geq c$. Finally, we can use $y = (-1, 8, -2)^T$ as a certificate of unfeasibility for LP1 because $y^T A = (5, 4, 3, 0, 0) \geq 0$ and $y^T c = -1 < 0$.

4: Consider an LP with constraints $Ax = b$, $x \geq 0$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & -1 \\ 2 & 2 & 1 & 1 & -1 \\ 1 & 2 & 1 & 0 & -2 \end{pmatrix} \ , \ b = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} .$$

Show that the LP is unfeasible, and find a certificate of unfeasibility.

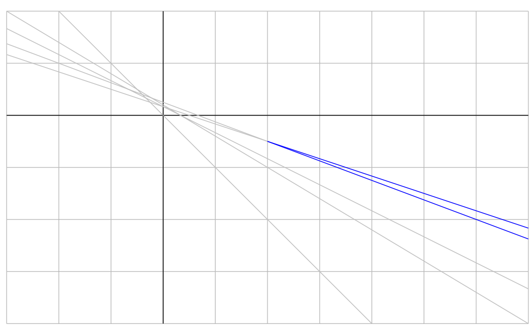
Solution: We need to find $y \in \mathbf{R}^3$ so that $y^T A \geq 0$ and $y^T b < 0$. We shall try to find y so that $y^T A \geq 0$ (or equivalently $A^T y \geq 0$) with $y^T b = -1$. We have

$$y^T b = -1 \iff y_1 - y_2 - 4y_3 = -1 \iff y = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

for some s, t, and we need

$$0 \leq A^T y = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & -1 & -2 \end{pmatrix} \left(\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} -1 \\ -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} + s \begin{pmatrix} 3 \\ 3 \\ 1 \\ 3 \\ -2 \end{pmatrix} + t \begin{pmatrix} 5 \\ 6 \\ 1 \\ 8 \\ -6 \end{pmatrix} ,$$

so (s, t) must satisfy $3s + 5t \geq 1$, $3s + 6t \geq 1$, $s + t \geq 0$, $3s + 8t \geq 2$ and $-2s - 6t \geq -1$. The set of feasible points (s, t) is shown below, outlined in blue.



One feasible choice is $(s, t) = (6, -2)$ and this choice gives

$$y = p + su + tv = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -2 \end{pmatrix} .$$

It is easy to verify that for $y = (-3, 6, -2)^T$ we have $y^T A = (7, 5, 4, 0, 1) \geq 0$ and $y^T b = -1$.

5: Consider the LP where we maximize $z = c^T x$ subject to $Ax = b$, $x \geq 0$ where

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 \end{pmatrix} \ , \ b = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} \ , \ c^T = \big(2, -1, 1, 2, 3\big)$$

Find an optimal solution $\overline{x}$ and a certificate of optimality.

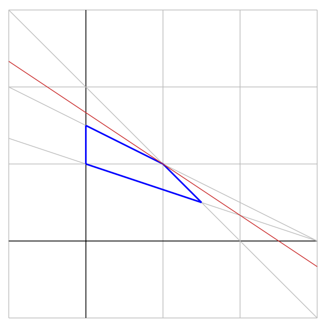
Solution: Let us solve this LP graphically. First we solve $Ax = b$. We have

$$\begin{aligned} \big(A\big|b\big) &= \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 4 \\ 2 & 1 & 1 & 2 & 2 & 5 \\ 2 & 0 & 1 & 1 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 4 \\ 0 & 3 & 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 & 1 & 2 \end{array}\right) \\ &\sim \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & -3 & -3 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & -3 & -3 \end{array}\right) \end{aligned}$$

so the solution is $x = p + su + tv$ where $p = (3, 2, -3, 0, 0)^T$, $u = (-1, -1, 1, 1, 0)^T$ and $v = (-2, -1, 3, 0, 1)^T$. We must maximize

$$z = c^T x = c^T\big(p + su + tv\big) = (c^T p) + (c^T)u + (c^T v)t = 1 + 2s + 3t$$

subject to the constraints $x_1 \geq 0, \cdots, x_5 \geq 0$ which we can rewrite as $-s - 2t \geq -3$, $-s - t \geq -2$, $s + 3t \geq 3$, $s \geq 3$ and $t \geq 0$. We draw a picture of the set of feasible points (s, t) which satisfy these constraints (outlined in blue) along with the level curve $z = \max$ (shown in orange).



We see from the picture that the maximum value of z occurs when $(s, t) = (1, 1)$, and then $z = 1 + 2s + 3t = 6$. The optimal solution is $\overline{x} = p + su + tv = (0, 0, 1, 1, 1)^T$. It remains to find a certificate of optimality for $\overline{x}$. We must find $y \in \mathbf{R}^3$ with $y^T b \geq c^T \overline{x}$ and $y^T A \geq c^T$ (or equivalently $A^T y \geq c$). We have $c^T \overline{x} = (2, -1, 1, 2, 3) \bullet (0, 0, 1, 1, 1) = 6$ and so

$$y^T b = c^T \overline{x} \iff 4y_1 + 5y_2 + 3y_3 = 6 \iff y = \begin{pmatrix} 3/2 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -5/4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3/4 \\ 0 \\ 1 \end{pmatrix}$$

for some s, t. Then we have

$$A^T y = \frac{1}{4} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \left(\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} \right) = \frac{1}{4} \left(\begin{pmatrix} 6 \\ 12 \\ 6 \\ 12 \\ 6 \end{pmatrix} + s \begin{pmatrix} 3 \\ -6 \\ -1 \\ -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 5 \\ -6 \\ 1 \\ -2 \\ 1 \end{pmatrix} \right)$$

and so $A^T y \geq c$ when (s, t) satisfies $3s + 5t \geq 12$, $-6s - 6t \geq -16$, $-s + t \geq -2$, $-2s - 2t \geq -4$ and $3s + t \geq 6$. As you can verify with the help of a picture, the only feasible point is given by $(s, t) = (2, 0)$, and then we have $y = \big(\frac{3}{2}, 0, 0\big)^T + 2\big(-\frac{5}{4}, 1, 0\big)^T = (-1, 2, 0)^T$. It is easy to verify that for $y = (-1, 2, 0)^T$ we have $y^T b = 6 = c^T \overline{x}$ and $y^T A = (3, 0, 1, 2, 3) \geq c^T$ and so y is indeed a certificate of optimality for $\overline{x}$.