CO 250 Intro to Optimization, Solutions to Assignment 1

1: Maximize and minimize $z = c^T x$ for $x \in \mathbf{R}^5$ subject to $Ax = b$ and $x \geq 0$, where

$$c = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 3 \\ -2 \end{pmatrix}, \ A = \begin{pmatrix} 1 & 4 & -3 & -2 & 3 \\ 1 & 3 & -2 & -1 & 2 \\ 1 & 2 & -2 & -3 & 3 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}.$$

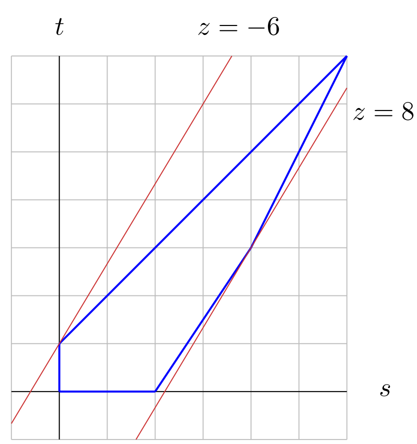
Solution: We solve $Ax = b$. We have

$$\begin{aligned}(A|b) &= \left(\begin{array}{ccccc|c} 1 & 4 & -3 & -2 & 3 & 3 \\ 1 & 3 & -2 & -1 & 2 & 4 \\ 1 & 2 & -2 & -3 & 3 & -1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 4 & -3 & -2 & 3 & 3 \\ 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 1 & 0 & 4 \end{array}\right) \\ &\sim \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & -1 & 7 \\ 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 6 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & -1 & 5 \\ 0 & 0 & 1 & 3 & -2 & 6 \end{array}\right)\end{aligned}$$

so the solution is $x = p + su + tv$ where $p = (1,5,6,0,0)^T$, $u = (1,-2,-3,1,0)^T$ and $v = (-1,1,2,0,1)^T$. We must optimize

$$z = c^T x = c \cdot (p + su + tv) = (c \cdot p) + (c \cdot u)s + (c \cdot v)t = -3 + 5s - 3t$$

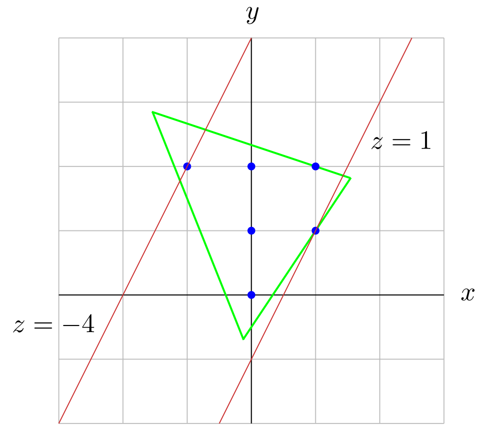
subject to the constraints $x_1 \geq 0, x_2 \geq 0, \cdots, x_6 \geq 0$ which we rewrite as $1 + s - t \geq 0$, $5 - 2s + t \geq 0$, $6 - 3s + 2t \geq 0$, $s \geq 0$ and $t \geq 0$. We draw a picture of the set of points (s,t) which satisfy these constraints (outlined in blue) along with the level curves $z = \min$ and $z = \max$ (shown in orange).



We see that the minimum value is $z = -6$ (which occurs when $(s,t) = (0,1)$) and the maximum value is $z = 8$ (which occurs when $(s,t) = (4,3)$).

2: Maximize and minimize $z = 2x - y$ for $x, y \in \mathbf{Z}$ (this is an integer program) subject to $x + 3y \leq 7$, $3x - 2y \leq 1$ and $5x + 2y \geq -2$.

Solution: The set of points (x, y) with $x, y \in \mathbf{R}$ which satisfy these constraints is outlined in green, and those with $x, y \in \mathbf{Z}$ are shown in blue. We also show the level curves $z = \min$ and $z = \max$ in orange.



We see that the maximum value is $z = 1$ (which occurs when $(x, y) = (1, 1)$) and the minimum is $z = -4$ (which occurs when $(x, y) = (-1, 2)$).

3: Maximize and minimize $z = x - 2y$ for $x, y \in \mathbf{R}$ subject to the non-linear constraints $x^2 + y^2 \leq 4x + 2y$ and $4y \leq x + xy$.

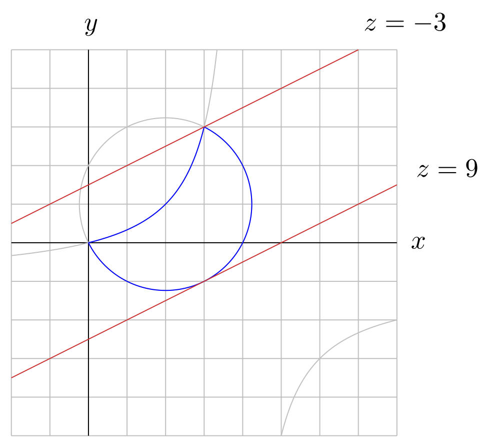
Solution: We complete the square to get

$$x^2 + y^2 \leq 4x + 2y \iff x^2 - 4x + y^2 - 2y \leq 0 \iff (x-2)^2 + (y-1)^2 \leq 5\,.$$

This is the region inside the circle of radius $\sqrt{5}$ centred at the point $(2, 1)$. Also, we have

$$4y \leq x + xy \iff y(4 - x) \leq x$$

and this is the region which lies between the two branches of the hyperbola $y = \dfrac{x}{4 - x}$. The set of all points (x, y) which satisfy these two constraints is shown below, outlined in blue, and the level sets $z = \max$ and $z = \min$ are shown in orange.



We see that the maximum value is $z = 5$ (which occurs when $(x, y) = (3, -1)$) and the minimum is $z = -3$ (which occurs when $(x, y) = (3, 3)$).

4: A company produces two products P_1 and P_2 which use two resources R_1 and R_2. They use 2 units of R_1 per unit of P_1 produced and 3 units of R_1 per unit of P_2 produced, and they have a total of 25 units of R_1 available. They use 1 unit of R_2 per unit of P_1 and 2 units of R_2 per unit of P_2, and they have a total of 16 units of R_2 available. They make a profit of 5 thousand dollars per unit of P_1 produced and 8 thousand dollars per unit of P_2 produced. How much would it be worth to purchase 4 more units of R_1?

Solution: Let x_1 be the amount of product P_1 produced and let x_2 be the amount of product P_2 produced. When they have 25 units of R_1 and 16 units of R_2 available, the company maximizes the profit $z = 5x_1 + 8x_2$ subject to the constraints

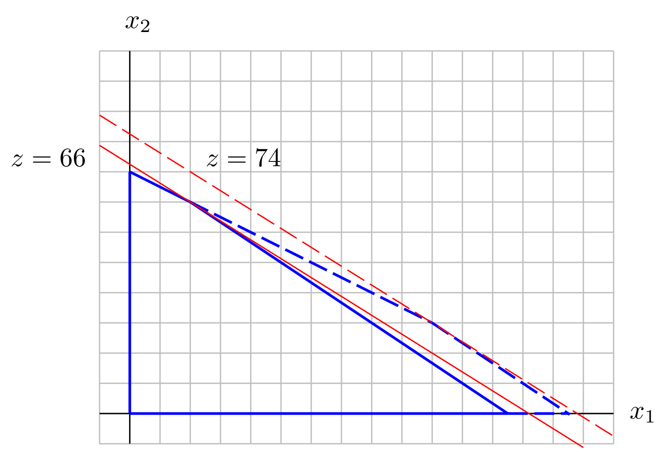
$$2x_1 + 3x_2 \leq 25 \ , \ x_1 + 2x_2 \leq 16 \, .$$

The feasible set is outlined below in blue and the level set $z = \max$ is shown in red. We see that the maximum profit occurs when $(x_1, x_2) = (2, 7)$ and the maximum profit is $z = 5 \cdot 2 + 8 \cdot 7 = 66$.

If the company buys an additional 4 units of R_1 then they will have 29 units of R_1 and 16 units of R_2. They will maximize the same objective function $z = 5x_1 + 8x_2$ subject to the new constraints

$$2x_1 + 3x_2 \leq 29 \ , \ x_1 + 2x_2 \leq 16 \, .$$

The new feasible set is outlined with a dashed blue line and the new level set $z = \max$ is shown as a dashed red line. We see that the new maximum profit occurs when $(x_1, x_2) = (10, 3)$, and the new maximum profit is $z = 5 \cdot 10 + 8 \cdot 3 = 74$. Since the maximum profit would increase from 66 to 74 thousand dollars, it would be worth 8 thousand dollars to purchase 4 more units of R_1.



5: Let P be the polygon $P = \left\{x \in \mathbf{R}^2 \middle| Ax \leq b\right\}$ where $A \in M_{n\times 2}(\mathbf{R})$ and $b \in \mathbf{R}^n$. Show that the problem of determining the maximum possible radius r for a circular disc which is contained in P can be formulated as a linear programming problem.

Solution: Let $a_1, \cdots, a_n$ be the row vectors of A. Then for $x \in \mathbf{R}^2$ we have

$$x \in P \iff Ax \leq b \iff a_i \cdot x \leq b_i \text{ for all } i\,.$$

For $\overline{x} \in \mathbf{R}^2$ and $r \geq 0$, let $D(\overline{x}, r)$ be the disc of radius r centred at $\overline{x}$, that is

$$D(\overline{x}, r) = \left\{x \in \mathbf{R}^2 \middle| |x - \overline{x}| \leq r\right\}.$$

We want to find $\overline{x}$ and r with $D(\overline{x}, r) \subseteq P$ such that r is as large as possible. For $\overline{x} \in \mathbf{R}^2$ and $r \geq 0$ we have

$$\begin{aligned} D(\overline{x}, r) \subseteq P &\iff \text{for all } i,\ a_i \cdot x \leq b_i \text{ for all } x \in D(\overline{x}, r) \\ &\iff \text{for all } i,\ \max\left\{a_i \cdot x \middle| x \in D(\overline{x}, r)\right\} \leq b_i \end{aligned}$$

Note that the point $x \in D(\overline{x}, r)$ which maximizes the dot product $a_i \cdot x$ is the point $x = \overline{x} + \frac{r}{|a_i|}\, a_i$. Indeed when $|u| \leq r$ (so that $x = \overline{x} + u \in D(\overline{x}, r)$) and θ is the angle between a_i and u, we have

$$a_i \cdot x = a_i \cdot (\overline{x} + u) = a_i \cdot \overline{x} + a_i \cdot u = a_i \cdot \overline{x} + |a_i||u| \cos\theta$$

and this is maximized when $|u| = r$ and $\theta = 0$, that is when $u = \frac{r}{|a_i|}\, a_i$. Thus we have

$$\begin{aligned} D(\overline{x}, r) \subseteq P &\iff \text{for all } i,\ \max\left\{a_i \cdot x \middle| x \in D(\overline{x}, r)\right\} \leq b_i \\ &\iff \text{for all } i,\ a_i \cdot \left(\overline{x} + \tfrac{r}{|a_i|}\, a_i\right) \leq b_i \\ &\iff \text{for all } i,\ a_i \cdot \overline{x} + r|a_i| \leq b_i\,. \end{aligned}$$

Thus we can solve the problem using the variables x_1, x_2, r, by maximizing $z = r$ subject to $a_i \cdot x + |a_i| r \leq b_i$ for $i = 1, 2, \cdots, n$. This is a linear programming problem.