1: Consider the LP where we maximize $z = c_0 + c^T x$ for $x \in \mathbf{R}^3$ subject to $Ax \le b$ and $x \ge 0$ where

$$c_0 = 1\ ,\ c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\ ,\ A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 6 \end{pmatrix} \text{ and } b = \begin{pmatrix} 6 \\ 8 \\ 18 \end{pmatrix}.$$

(a) Draw an accurate picture of the feasible set in $\mathbf{R}^3$.

(b) Find the exact coordinates of all the vertices of the feasible set.

(c) Solve the LP.

2: (a) Consider the LP where we *minimize* $z = x_1 - 2x_2 - x_3 - 5$ subject to the constraints $x_1 + 2x_2 - x_3 = 3$, $2x_1 - 5x_2 + 3x_3 = 1$, $x_1 + 3x_2 + 2x_3 \le 4$, $-x_1 - x_2 + 3x_3 \ge -1$ and $x_2 - 2x_3 \ge 1$. Convert this to an equivalent LP in SEF.

(b) Consider the LP where we maximize $z = c_0 + c^T x$ for $x \in \mathbf{R}^5$ subject to the constraints $Ax = b$ and $A'x \ge b'$ where $c_0 = 6$ and

$$c = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 1 \end{pmatrix},\ A = \begin{pmatrix} 1 & 2 & 1 & 3 & 2 \\ 2 & 4 & 1 & 2 & 3 \end{pmatrix},\ A' = \begin{pmatrix} 1 & 2 & 0 & 1 & 4 \\ 1 & 3 & 1 & 2 & 1 \end{pmatrix},\ b = \begin{pmatrix} 1 \\ 5 \end{pmatrix},\ b' = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

By first solving the equation $Ax = b$, convert this to an equivalent LP where we must maximize $z = \widetilde{c}_0 + \widetilde{c}^T \widetilde{x}$ for $\widetilde{x} \in \mathbf{R}^8$ subject to $\widetilde{A}\,\widetilde{x} = \widetilde{b}$, $\widetilde{x} \ge 0$, where $\widetilde{c} \in \mathbf{R}^8$ and $\widetilde{b} \in \mathbf{R}^2$.

3: Consider the following three LPs where we maximize $z = c^T x$ subject to $Ax = b$, $x \ge 0$.

$$\text{LP1}: A = \begin{pmatrix} 1 & 2 & -1 & -4 & 6 \\ 1 & 1 & 0 & -1 & 2 \\ 1 & 1 & -1 & -2 & 5 \end{pmatrix},\quad b = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix},\quad c^T = (3, 1, 2, -1, 1)$$

$$\text{LP2}: A = \begin{pmatrix} 1 & 1 & -1 & -2 & 3 \\ 0 & 1 & 1 & -1 & -2 \\ 1 & 0 & -1 & 0 & 2 \end{pmatrix},\quad b = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix},\quad c^T = (2, 0, -1, -2, 3)$$

$$\text{LP3}: A = \begin{pmatrix} 1 & 1 & -1 & -2 & 3 \\ -1 & 1 & 0 & -3 & -2 \\ -2 & 1 & 1 & -3 & -6 \end{pmatrix},\quad b = \begin{pmatrix} 2 \\ -5 \\ -9 \end{pmatrix},\quad c^T = (2, 0, 1, 4, -3)$$

One of these LPs is infeasible, one is unbounded, and one has an optimal solution. Determine which is which by selecting suitable certificates from amongst the following vectors

$$\overline{x}^T, y^T \in \left\{ (1,0,5,1,2)\ ,\ (0,3,0,2,1)\ ,\ (3,0,5,2,3)\ ,\ (-1,8,-2)\ ,\ \tfrac{1}{4}(16,-39,23) \right\}$$

4: Consider an LP with constraints $Ax = b$, $x \geq 0$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & -1 \\ 2 & 2 & 1 & 1 & -1 \\ 1 & 2 & 1 & 0 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}.$$

Show that the LP is unfeasible, and find a certificate of unfeasibility.

5: Consider the LP where we maximize $z = c^T x$ subject to $Ax = b$, $x \geq 0$ where

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}, \quad c^T = \big(2, -1, 1, 2, 3\big)$$

Find an optimal solution $\overline{x}$ and a certificate of optimality.