Notes prepared by
Stanley Burris
February 10, 1997

Spectrum of a sentence

In 1952 Scholz asked about the possibilities for the set of sizes of the finite models of a first-order sentence.

DEFINITION 1 Given a first-order sentence φ define the *spectrum* of φ to be the set $\{|\mathrm{A}| : \mathbf{A} \models \varphi,\ \mathrm{A} \text{ finite}\}$.

Let us look at some examples of spectra.

EXAMPLE 2 The set of powers of primes is a spectrum: take the statement defining fields.

EXAMPLE 3 The set of even numbers is a spectrum: take the statement $\forall x (x \not\approx f(x) \wedge x \approx f(f(x)))$.

An interesting set of models is obtained by using a modification of Dedekind's approach to the natural numbers, namely let our language be $\{+, \times, \leq, ', 2, b, c\}$ and let φ_0 be the conjunction of the statements:

- $\forall x\ x \leq x$
  $\forall x \forall y\ x \leq y \wedge y \leq x\ \rightarrow\ x \approx y$
  $\forall x \forall y \forall z\ x \leq y \wedge y \leq z\ \rightarrow\ x \leq z$
  [$\leq$ is a linear ordering]

- $\forall x\ 2 \leq x \wedge x \leq c$
  [2 is the smallest element, c the largest element]

- $x < c \rightarrow x < x'$
  [$'$ is strictly increasing on the elements less than c]

- $c' \approx c \wedge b < c \wedge b' \approx c$

- $\forall x \forall y\ y \leq x \vee x' \leq y$
  [there are no elements between x and x']

- $\forall x\ x + 2 \approx x''$
- $\forall x \forall y\ x + y' \approx (x + y)'$
- $\forall x \forall y\ x \times 2 \approx x + x$
- $\forall x \forall y\ x \times y' \approx (x \times y) + x$

One can see that for each natural number $n > 1$ there is one model (up to isomorphism) of size n, and it looks like the numbers greater than 1 with the usual ordering $\leq$ and operations $+, \times, '$, with the numbers $\geq n+1$ collapsed into the single number $n+1$, which is the element labelled c, and n is labelled b.

EXAMPLE 4 The set of composite numbers is a spectrum: take $\varphi_0 \wedge \psi$ where ψ is $\exists x \exists y\, (x < b \wedge y < b \wedge x \times y \approx b)$.

EXAMPLE 5 The set of prime numbers is a spectrum: take $\varphi_0 \wedge \neg\psi$, ψ as in example 4

One can easily show that the union or intersection of two spectra is again a spectrum (make the languages of φ and ψ disjoint and consider $\varphi \vee \psi$ and $\varphi \wedge \psi$). In 1956 Asser looked at the question of whether the complement of a spectrum is again a spectrum — the problem is still open. A fascinating result was proved by Jones and Selman in 1974 when they showed that a subset of N is a spectrum iff it can be accepted by some nondeterministic Turing machine in time $2^{O(n)}$. From this it follows that if NP is closed under complements, i.e., NP = co-NP, then spectra are also closed under complements. Consequently to show that spectra are not closed under complements is at least as difficult as showing NP $\neq$ co-NP, and hence P $\neq$ NP.

Exercises

Problem 1 Show that the union or intersection of two spectra is again a spectrum.

Problem 2 Show that $\{n^2 : n \in N\}$ is a spectrum.

Problem 3 Show that $\{n^2 + 1 : n \in N\}$ is a spectrum.

Problem 4 Show that $\{2^n + 1 : n \in N\}$ is a spectrum.

Problem 5 Show for a fixed n that $\{p^n : p \text{ a prime}\}$ is a spectrum.

Problem 6 $\star$ Show that $\{F_n : n \in N,\ F_n{=}n^{\text{th}} \text{ Fermat prime}\}$ is a spectrum.

Problem 7 $\star$ Show that $\{2^x 3^y 5^z 7^w : x^w + y^w = z^w\}$ is a spectrum.