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February 10, 1997

Principia Mathematica: Whitehead and Russell

A.N. Whitehead (1861–1947) and B. Russell (1872–1970)
1910–1913 - Principia Mathematica I–III

Russell met Peano at the 1900 International Congress of Mathematicians in Paris, and was captivated by Peano's work on foundations. And, starting in 1900, he was studying the *Grundgesetze I* of Frege. This led to his discovery of the famous contradiction in Frege's system in June, 1901, while writing his *Principles of Mathematics (1903)*. Nonetheless, Russell and Whitehead, who started their joint work on foundations in 1900, would carry out the program of Frege to a significant extent, namely the seamless development of mathematics from a few clearly stated axioms and rules of inference in pure logic. However they opted for the more modern notation of Peano instead of Frege's Begriffsschrift. Their work, *Principia Mathematica,* filled three volumes, almost 2,000 pages, and appeared in the years 1910–1913. Their approach was essentially that of Frege, to define mathematical entities, like numbers, in pure logic and then derive their fundamental properties. Indeed their definition of natural numbers was essentially that of Frege, but unlike him, they opted to avoid the philosophical aspects and justifications. In the preface they say

> We have avoided both controversy and general philosophy, and made our statements dogmatic in form $\cdots$.
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> The general method which guides our handling of logical symbols is due to Peano. His great merit consists not so much in his definite logical discoveries nor in the details of his notations (excellent as both are), as in the fact that he first showed how symbolic logic was to be freed from its undue obsession with the forms of ordinary algebra, and thereby made it a suitable instrument for research $\cdots$.
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> In all questions of logical analysis, our chief debt is to Frege.

The main innovation of *Principia Mathematica* was to introduce a stratification of Frege's formulas into types, and to use this to restrict which of

Frege's formulas would be permitted in their logic. The key idea was that a formula φ could not be substituted for a variable x in a formula ψ unless the variable x was of the appropriate type. Thus, returning to Frege's troublesome theorem

$$P(y) \leftrightarrow y \in \{x : P(x)\},$$

the types restriction would prevent the substitution of $x \notin x$ for *both* of the variables P and y since P is of a higher type than its argument y. Indeed all the known paradoxes were avoided by using types.

Having salvaged Frege's logic, they proceeded to develop some of the elementary theorems of mathematics, covering far more ground than Frege — however we note that they quickly adopted the convention of leaving out easy steps of proofs — and, at the same time, falling far short of the list of theorems in Peano's *Formulario.* Let us briefly sketch the topics covered in *Principia Mathematica:*

Vol. I: Axioms and rules of inference for their higher order logic; elementary results on classes and binary relations (e.g., the study of union, intersection, domain, range, one-to-one, onto, converse, composition, restriction); the definition of the numbers 1 (p. 347) and 2; a discussion of Zermelo's Well-ordering Theorem and the Axiom of Choice, and choice functions; the Schröder-Bernstein theorem; the transitive closure of a relation.

Vol II: Cardinal numbers and their arithmetic; finite numbers; the arithmetic of binary relations; linear orderings; Dedekind orderings; limit points; continuous functions.

Vol III: Well-orderings; equivalence of the Axiom of Choice with the Well-ordering Axiom; the $\aleph$'s; dense orderings; orderings like the rationals; orderings like the reals; the integers, rationals, and reals; measurement; measurement modulo a quantity.

A fourth volume, on geometry, never appeared. Although the above topics may look like a small fragment of mathematics, nonetheless Russell and Whitehead had carried the dream of Frege far enough, and in a transparent enough symbolism, that the possibility of developing all of mathematics from a few axioms and rules was made clear. Future developments would focus on the best way to do this, plus efforts to guarantee that one would not find any contradictions. Most important for future developments was the fact that the great leader of mathematics Hilbert would become heavily involved in mathematical logic.

So by 1931 Gödel could boast of two formal systems which, with a few axioms and rules, could encompass all known mathematics, namely Principia Mathematica and the Zermelo-Fraenkel axiom system.

Nonetheless, if one of these systems is consistent, then Gödel showed it would not be strong enough to prove all first-order truths about the non-negative integers (using $+$ and $\times$ as the operation symbols). And in 1936 Church would show that Peano Arithmetic, if consistent, is undecidable (Rosser improved this in 1936 to show any consistent extension of Peano Arithmetic is undecidable).