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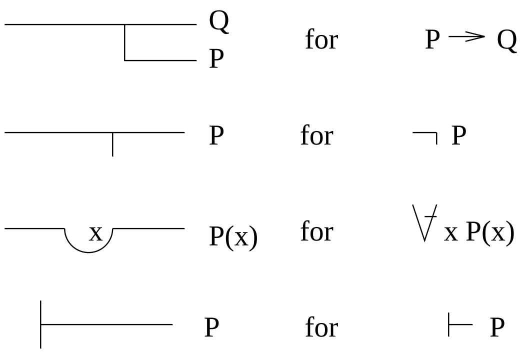
Concept Script: Frege

Gottlieb Frege (1848–1925)
1879 - Begriffsschrift
1884 - Grundlagen der Arithmetik
1893 - Grundgesetze der Arithmetik I.
1903 - Grundgesetze der Arithmetik II.

Frege initiated an ambitious program to use a precise notation which would help in the rigorous development of mathematics. Although his efforts were almost entirely focused on the natural numbers, he discussed possible applications to geometry, analysis, mechanics, physics of motion, and philosophy.

The precise notation of Frege was introduced in *Begriffsschrift* (Concept Script) in 1879. This was a two-dimensional notation whose powers he compared to a microscope. The framework in which he set up his *Begriffsschrift* was quite simple — we live in a world of objects and concepts, and we deal with statements about these in a manner subject to the laws of logic. Thus Frege had only one model in mind, the real world. Let us refer to this as the *absolute universe.* From this he was going to distill the numbers and their properties. The absolute universe approach to mathematics via logic was dominant until 1930 — we see it in the work of Whitehead and Russell (1910–1913).

His formal system with two-dimensional notation had the universal quantifier, negation, implication, predicates of several variables, axioms for logic, and rules of inference. The explicit universal quantifier, predicates of several variables and the rules of inference were new to formal systems!



Begriffschrift notation

The rule of inference and axioms found at the beginning of Begriffsschrift are

RULE of INFERENCE

$$\frac{P \to Q,\ P}{Q}$$ ←modus ponens

AXIOMS	
$P \to (Q \to P)$	$\neg\neg P \to P$
$(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$	$P \to \neg\neg P$
$(P \to (Q \to R)) \to (Q \to (P \to R))$	$x \approx y \to (P(x) \to P(y))$
$(P \to Q) \to (\neg Q \to \neg P))$	$x \approx x$
$\forall x P(x) \to P(x)$	

Higher order quantification was also permitted in the Begriffsschrift, but the axioms and rules for working with such were not presented.

The two-dimensional notation, the lack of new mathematical results, and the tedious work required ensured that *Begriffsschrift* would be almost totally ignored. Nonetheless there were major contributions in this paper, namely

i. An elegant propositional logic based on $\to$ and $\neg$,

ii. a notation for universally quantified variables,

iii. relation symbols of several variables,[1]

and these were incorporated in

iv. a powerfully expressive higher order logic, the likes of which had never been seen before.

Frege's attempt to set up the natural numbers in logic is based on what he calls his definition of a *sequence* — this is his main application of his logic to mathematics. Although his ultimate goal is to abstractly describe the sequence of natural numbers N with the usual ordering $\leq$, he only succeeds in describing a broader class of "sequences".

The crucial point in his work is that from a notion of *successor* he wants to be able to capture the notion of "y follows x" without using the obvious "for some integer n, y is the n^{th} element after x" — for otherwise his development of the natural numbers will be circular.

In modern notation he proceeds as follows. A property P is *hereditary* for a binary relation r, written $\text{Hered}(P, r)$, if

$$\forall x \forall y\, [(P(x) \land r(x, y)) \to P(y)]$$

holds. The relation r is *one-one*, written $E(r)$, if

$$\forall x \forall y \forall z\, [r(x, y) \land r(x, z)) \to y \approx z].$$

Given any binary r he defines a binary relation $\leq$ by

$$x \leq y \qquad \text{iff} \qquad x \approx y \lor \forall P\, [\text{Hered}(P, r) \to \forall z\, ((r(x, z) \land P(z)) \to P(y))].$$

Then the final results of Begriffsschrift are the transitivity and comparability properties of $\leq$:

$$\forall x \forall y \forall z\, [(x \leq y \land y \leq z) \to x \leq z] \tag{1}$$

$$E(r) \to \forall x \forall y (x \leq y \lor y \leq x). \tag{2}$$

Frege does not specify a particular relation r which will lead to a sequence like the natural numbers — this will first appear in Dedekind, 1888, where r is selected to be one-one, not onto, function from a domain to itself. Dedekind will use only one property P, namely select any a from the domain,

[1]Frege used the word 'function' where we now use the word 'relation'. This was again adopted by Hilbert and Ackermann in the second edition of their book (1938). Unfortunately we also use the words 'function symbol' in modern logic, but with quite a different meaning, namely such will be interpreted as a function on a domain to itself.

but not from the range, of r, and let $P(x)$ be "x is in the subuniverse generated by a". One can of course define the "subuniverse generated by a" without reference to n-fold applications of r to a; and it is a hereditary property. Dedekind, in a later introduction to his work, will state that in his development of the natural numbers he was unaware of Frege's work of 1879.

Frege's later work on the foundations of arithmetic abandons this work on sequences and picks up Cantor's definition of cardinal number based on one-one correspondences.

Obviously one has only begun to investigate number systems at the end of *Begriffsschrift*. Indeed, numbers have not even been defined. The reviews ranged from mediocre to negative. In particular Schröder thought Boole's logical system was far superior — Boole used the arithmetic notation for plus and times, and had marvelled at how much the laws of logic were like the laws of arithmetic. Schröder showed how much easier it was to write out the propositional part of Frege's work in Boole's notation.

In 1884 Frege published his second book on his approach to numbers, *Grundlagen der Arithmetik* (Basic Principles of Arithmetic). However this time he tried for popular appeal by omitting any scientific notation and using prose to explain his ideas. Although he was quite content with the foundations of geometry,[2] his theme that no one had provided a decent foundation for numbers was discussed at length. He presents several explanations of the nature of numbers which could be found in the literature, pointing out the shortcomings of each in turn.

In the last part of the book he proposed to solve the foundational question by showing that numbers can be obtained from pure logic. His main tool was the then well-known notion of one-to-one correspondence. Using this he defined the *cardinal number* (Anzahl) of a property P as the collection of all properties Q such that the class defined by Q can be put in one-to-one correspondence with the class defined by P. Then he defined 0 to be the cardinal number of the property $x \not\approx x$. Next he defined what it means for one number *to immediately follow* another, defined the number 1 to be the cardinal number of the property $x \approx 0$, and stated some elementary theorems.

At the end of *Grundlagen,* Frege says that with his notation and definitions it will be possible to carry out a development of the basic laws of

[2] In a letter to Hilbert at a later date he said there was no reason to prove the axioms of geometry consistent because they were true.

arithmetic without gaps. *Grundlagen* received three reviews, all negative (one was by Cantor). Nonetheless Frege turned to the writing of his final masterpiece, the two volumes of *Grundgesetze der Arithmetik* (Basic Laws of Arithmetic). As the work neared completion he had difficulty finding a publisher because of the poor showing of his previous books on this subject. Finally he found a publisher who agreed to publish *Grundgesetze I,* and would publish Vol. II provided Vol. I was successful.

Grundgesetze I appeared in 1893. Again Frege emphasizes the need for a rigorous development of numbers. And as usual he finds fault with what has been offered by others. He says that Dedekind's *Was sind und was sollen die Zahlen?* is full of gaps. We quote: (*Grundgesetze*, p. VII)

> To be sure this [Dedekind's] brevity is attained only because a great deal is really not proved at all $\cdots$ an inventory of the logical or other laws which he takes as basic is nowhere to be found.

Also he says that Schröder's *Algebra der Logik* (1890–1910) is more concerned with techniques and theorems than with foundations. As to Frege's standards and goals we can do no better than quote from the foreword of *Grundgesetze I*, p. VI:

> The idea of a strictly scientific method in mathematics, which I have attempted to realize, and which might indeed be named after Euclid, I should like to describe as follows. It cannot be demanded that everything be proved, because that is impossible; but we can require that all propositions used without proof be expressly declared as such, so that we can see distinctly what the whole structure rests upon. After that we must try to diminish the number of primitive laws as far as possible, by proving everything that can be proved. Furthermore, I demand — and in this I go beyond Euclid — that all methods of inference employed be specified in advance $\cdots$

In *Grundgesetze I* Frege introduces some items which did not appear in *Begriffsschrift,* namely

i. True and False[3]

[3] Frege describes the truth value of $P \to Q$ for each of the four possible truth values of P, Q. Thus we almost have the first *truth table;* but there is no table, just a verbal description. Truth tables are usually attributed to Wittgenstein who was well versed in Frege's work, and had gone to study with Russell at Frege's suggestion.

ii. $\{x : P(x)\}$ for the class determined by the property P (his notation was like x'$P(x)$),

iii. $\backslash x\{y : P(y)\}$ for the unique element satisfying $P(y)$, if such exists; and otherwise this signifies just $\{y : P(y)\}$, and

iv. $x \in y$ for x is a member of the class y (his notation was $x \frown y$).

In *Grundgesetze I* he only needs the single rule of inference modus ponens. For the axioms he takes, in addition to those for the propositional calculus, the following:

i. $\forall x\, P(x) \to P(x)$,

ii. $\forall P\, P(x) \to P(x)$,

iii. $x \approx y \to \forall P\,(P(x) \to P(y))$

iv. $\{x : P(x)\} \approx \{x : Q(x)\} \leftrightarrow \forall x\,(P(x) \leftrightarrow Q(x))$, and

v. $x \approx \backslash y\{y : x \approx y\}$.

Grundgesetze I had only two reviews, again neither very favorable. Peano was one of the reviewers, and he used the review to point out the advantages of his own approach to number systems; in particular he thought the fact that he used fewer primitives (three) than Frege was a big plus for his system. In 1896 Peano received a detailed response from Frege, pointing out that in fact there were at least nine primitive notions in Peano's system, not to mention the incompleteness and confusion at certain points. Peano was able to incorporate some of Frege's suggestions into Vol. II of Formulario (1899), and, as requested, he also published Frege's letter with a recanting of statements in his review of *Grundgesetze I.*

Around 1900 the young Bertrand Russell was studying the work of Frege. Frege lack of serious study of his foundations of numbers, but this was to change. Frege was very confident about his work, with perhaps one exception. We quote from the foreword of *Grundgesetze I*, p. VII:

> A dispute can arise, as far as I can see, only with my basic law concerning the domain of definition (V), which logicians perhaps have not yet expressly enunciated, and yet is what people have in mind, for example, when they speak of extensions of concepts. I hold that it is a law of pure logic. In any event, the place is noted where a decision must be made.

The questionable law[4] (V) (*Grundgesetze I*, p. 36) was

$$\{x : P(x)\} \approx \{x : Q(x)\} \leftrightarrow \forall x\,(P(x) \leftrightarrow Q(x)).$$

One of the propositions Frege derived using (V) was (*Grundgesetze I*, p. 117)

$$P(y) \leftrightarrow y \in \{x : P(x)\}.$$

In June of 1901 Russell discovered that Frege's logical system led to contradictions, for if one lets $P(x)$ be the property $x \notin x$ and lets y be $\{x : x \notin x\}$ we have[5]

$$\{x : x \notin x\} \notin \{x : x \notin x\} \leftrightarrow \{x : x \notin x\} \in \{x : x \notin x\}.$$

Russell communicated this to Frege in June, 1902, when *Grundgesetze II* was just about ready to appear (Frege had not found a publisher, so he was paying for the publication out of his own pocket). Working on the contradiction during the summer he was able to compose an appendix, pointing out the flaw in his system, and suggesting a remedy. (Years later it was discovered that the remedy reduced the universe to one object!)

The first 160 pages of *Grundgesetze II* (1903) are devoted to criticism of existing approaches to the real numbers. Frege discusses briefly defining the reals using the integer part plus a dyadic expansion for the remainder. He does not do any technical work with this, and indeed says that ordered pairs of integers and dyadic expansions will not be the reals. His final theorem in *Grundgesetze II* is a proof of the commutative law for addition for so called positive classes (which are defined to be classes having several of the properties of the positive reals). His final statement is to say that it remains to find a suitable positive class to develop the reals. So, in the end, Frege has no real numbers to show.

After 500 pages of *Grundgesetze I/II* and 689 propositions one could have hoped for more than the commutative law. Nonetheless Frege made a major contribution to the precision of presentation. Nowadays, when studying formal systems, after seeing how formalization is actually carried out we are usually content to skip over as many of the details as possible, quite the opposite of Frege's approach.

[4] In Frege's book it was written

$$\vdash\!\!\!\longrightarrow (\varepsilon\,'f(\varepsilon) = \alpha\,'g(\alpha)) = (\longrightarrow\!\!\overset{a}{\smile}\!\!\longrightarrow f(a) = g(a)).$$

[5] A slight variation on this applies to $\varphi(P) = \neg P(P)$, namely one gets $\varphi(\varphi) \leftrightarrow \neg\varphi(\varphi)$.

Whitehead & Russell picked up on the work of Frege; they solved the immediate contradictions by *typing* the predicates and variables. Thus one could not use $P(P)$ in their system because P must be of higher type than its argument, nor would one be allowed to use $x \in x$.

EXERCISES

Problem 1 Give a proof of $P \to P$ from the axioms of Frege, in the notation of Frege.

Problem 2 Prove the transitivity of $\leq$ claimed in 1

Problem 3 Prove the comparability claimed in 2

Problem 4 Show that for f a unary function from a domain D to itself, and for fixed $a \in D$, the property $P(x)$ which says that "x is in the subuniverse generated by a" is a hereditary property of f.