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A proof of Boole's Elimination Theorem

THEOREM I.4.7 OF **LMCS**

This theorem is stated on page 24 of **LMCS**. Boole's proofs, as explained in the historical remarks, were pretty 'wild'. Here is a modern version.

THEOREM [Elimination Theorem] The most general equation $F(\vec{B}) \approx 0$ that follows from $E(\vec{A}, \vec{B}) \approx 0$ is obtained by setting

$$F(\vec{B}) = E(1,1,\cdots,1,\vec{B})E(0,1,\cdots,1,\vec{B})\cdots E(0,0,\cdots,0,\vec{B}),$$

or, more briefly,

$$F(\vec{B}) = \bigcap_{\sigma} E(\sigma, \vec{B}),$$

where σ ranges over sequences of 1's and 0's.

Proof. First we show $F(\vec{B}) \approx 0$, for the stated choice of F. By expanding E about A_1 (see Problem 4.4 on page 25) we have

$$E(\vec{A}, \vec{B}) \approx E(1, \widehat{A}, \vec{B})A_1 \cup E(0, \widehat{A}, \vec{B})A_1',$$

where $\widehat{A} = A_2 \cdots A_m$. Then from $E(\vec{A}, \vec{B}) \approx 0$ follows

$$\begin{aligned} E(1, \widehat{A}, \vec{B})A_1 &\approx 0 \\ E(0, \widehat{A}, \vec{B})A_1' &\approx 0, \end{aligned}$$

and thus

$$\begin{aligned} E(1, \widehat{A}, \vec{B})E(0, \widehat{A}, \vec{B})A_1 &\approx 0 \\ E(1, \widehat{A}, \vec{B})E(0, \widehat{A}, \vec{B})A_1' &\approx 0. \end{aligned}$$

Taking the union of these two gives

$$E(1, \widehat{A}, \vec{B})E(0, \widehat{A}, \vec{B})(A_1 \cup A_1') \approx 0,$$

and thus

$$E(1,\widehat{A},\vec{B})E(0,\widehat{A},\vec{B}) \quad \approx \quad 0.$$

Now repeat the above steps, except this time expand about A_2, using the last equation which has the form $E'(\widehat{A},\vec{B}) \approx 0$, to obtain

$$(E(1,1,\widetilde{A},\vec{B})E(0,1,\widetilde{A},\vec{B}))(E(1,0,\widetilde{A},\vec{B})E(0,0,\widetilde{A},\vec{B})) \quad \approx \quad 0,$$

where $\widetilde{A} = A_3,\cdots,A_m$. Continuing one arrives at the desired conclusion that $F(\vec{B})$, as defined above, is 0.

For the converse suppose $H(\vec{B}) \approx 0$ follows from $E(\vec{A},\vec{B}) \approx 0$. Let $K(\vec{B})$ be any $\vec{B}$-constituent of H. Then for any $\vec{A}$-constituent $L(\vec{A})$ we have $L(\vec{A})K(\vec{B})$ is an $\vec{A},\vec{B}$-constituent of H. But then, by Theorem 4.5, LK must be an $\vec{A},\vec{B}$-constituent of E as $H \approx 0$ follows from $E \approx 0$. Then $E(\sigma_L,\vec{B})$ has K as a $\vec{B}$-constituent, i.e., $E(\sigma_L,\vec{B}) \approx K \cup \cdots$. So K is a $\vec{B}$-constituent of F as F is defined to be the intersection over the L's of the $E(\sigma_L,\vec{B})$'s.

So every $\vec{B}$-constituent of H is also a $\vec{B}$-constituent of F. This means, by Theorem 4.5, that $H \approx 0$ follows from $F \approx 0$, so $F \approx 0$ is a more general conclusion than $H \approx 0$. ■