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February 10, 1997

Downward Löwenheim-Skolem theorem

In 1915 Löwenheim gave a nearly complete proof of the fact that if a first-order formula φ has a model, then it has a finite or countable model. A tidy, correct version of this was given by Skolem in 1922. We will give the more general version for arbitrary sets of formulas and arbitrary languages.[1]

DEFINITION 1 Given a language $\mathcal{L}$ and a set of variables X let $\kappa = \max(\aleph_0, |\mathcal{L}|, |X|)$. κ is called the *power* of the language $\mathcal{L}$.

THEOREM 2 [DOWNWARD LÖWENHEIM-SKOLEM] If Σ is a set of $\mathcal{L}$-formulas which is satisfiable, and the power of the language $\mathcal{L}$ is κ, then Σ can be satisfied by a model $(\mathbf{S}, \lambda)$ where $|\mathrm{S}| \leq \kappa$.

PROOF. Let $(\mathbf{S}, \lambda) \models \Sigma$, and let Σ' be a Skolemization of Σ. Then expand $(\mathbf{S}, \lambda)$ to a model $(\mathbf{S}', \lambda)$ of Σ'. Let S'' be the subset of S generated by the constants and the range of λ. S'' is closed under the operations of $\mathbf{S}'$; so let $\mathbf{S}''$ be the structure obtained by restricting the functions and relations of $\mathbf{S}'$ to S''. One sees that $(\mathbf{S}'', \lambda) \models \Sigma'$, and that $|\mathrm{S}''| \leq \kappa$. By reducing back to the original language we have the desired model of Σ. ∎

A remarkable consequence of this result is the existence of small models of axiomatic first-order set theory (provided any model exists).

COROLLARY 3 [SKOLEM] If ZF (or ZFC or vNBG) set theory has a model, it must have a countable model.

This seems to pose an obvious problem, because in such set theories one can prove that the power set of a set has strictly greater cardinality than the set — but if we have a countable model then all of our sets are countable. This is called Skolem's paradox.

EXERCISES

Problem 1 Given an arbitrary group $\mathbf{G}$ show that there is a countable subgroup $\mathbf{H}$ such that $\mathrm{Th}(\mathbf{H}) = \mathrm{Th}(\mathbf{G})$.

[1] Mal'cev is credited for extending first-order languages to arbitrary cardinalities.