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Was sind und was sollen die Zahlen?: Dedekind

Richard Dedekind (1831–1916)
1872 - Stetigkeit und irrationale Zahlen
1888 - Was sind und was sollen die Zahlen

Let us recall that by 1850 the subject of analysis had been given a solid footing in the real numbers — infinitesimals had given way to small positive real numbers, the ε's and δ's. In 1858 Dedekind was in Zürich, lecturing on the differential calculus for the first time. He was concerned about his introduction of the real numbers, with crucial properties being dependent upon the intuitive understanding of a geometrical line.[1] In particular he was not satisfied with his geometrical explanation of why it was that a monotone increasing variable, which is bounded above, approaches a limit. By November of 1858 Dedekind had resolved the issue by showing how to obtain the real numbers (along with their ordering and arithmetical operations) from the rational numbers by means of cuts in the rationals — for then he could prove the above mentioned least upper bound property from simple facts about the rational numbers. Furthermore, he proved that applying cuts to the reals gave no further extension.

These results were first published in 1872, in *Stetigkeit und irrationale Zahlen*. In the introduction to this paper he points out that the real number system can be developed from the natural numbers:

> I see the whole of arithmetic as a necessary, or at least a natural, consequence of the simplest arithmetical act, of counting, and counting is nothing other that the successive creation of the infinite sequence of positive whole numbers in which each individual is defined in terms of the preceding one.

In a single paragraph he simply states that, from the act of creating successive whole numbers, one is led to the concept of addition, and then to

[1] Recall that in geometry some mathematicians had already taken efforts to eliminate the dependence of the proofs on drawings.

multiplication. Then to have subtraction one is led to the integers. Finally the desire for division leads to the rationals. He seems to think that the passage through these steps is completely straight-forward, and he does not give any further detail.

Given the rationals he comes to the conclusion that what is missing is *continuity*, where continuity for him refers to the fact that you cannot create new numbers by cuts. By applying cuts to the rationals he gets the reals, lifts the operations of addition, etc., from the rationals to the reals, and then shows that by applying cuts to the reals no new numbers are created.

In his penetrating 1888 monograph Dedekind returns to numbers. The nature of numbers was a topic of considerable philosophical interest in the latter half of the 1800's — we have already said much about Frege on this topic. In 1887 Kronecker published Begriff der Zahl, in which he does rather little of technical interest, but he does quote an interesting remark which Gauss made in a letter to Bessel in 1830. Gauss says that numbers are distinct from space and time in that the former are a product of our mind. Dedekind picks up on this theme in the introduction to his monograph when he says

> In view of this freeing of the elements from any other content (abstraction) one is justified in calling the numbers a free creation of the human mind.

This seems to contrast with Kronecker's later remark:

> God made the natural numbers. Everything else is the work of man.

Regarding the importance of the natural numbers, Dedekind says that it was well known that every theorem of algebra and higher analysis could be rephrased as a theorem about the natural numbers[2] — and that indeed he had heard the great Dirichlet make this remark repeatedly (*Stetigkeit*, p. 338). Dedekind now proceeds to give a rigorous treatment of the natural numbers, and this will be far more exacting than his cursory remarks of 1872 indicated. Actually Dedekind said he had plans to do this around 1872, but

[2]For example, the Riemann hypothesis is equivalent to the following statement about the reals (μ is the Möbius function):

$$\forall \varepsilon > 0 \exists x \forall y \left(y > x \Longrightarrow \left(|\sum_{n=1}^{y} \mu(n)| < y^{1/2+\varepsilon} \right) \right),$$

and this can in turn be reduced to a statement about the natural numbers.

due to increasing administrative work he had managed, over the years, to jot down only a few pages. Finally, in 1888, he did finish the project, and published it under the title *Was sind und was sollen die Zahlen?*

Dedekind starts by saying that *objects* (Dinge) are anything one can think of; and collections of objects are called *classes* (Systeme), which are also objects. He takes as absolutely fundamental to human thought the notion of a *mapping*. He then defines a *chain* (Kette) as a class A together with a mapping $f : A \to A$, and proves that *complete induction* holds for chains, i.e., if A and f are given, and if B is a set of generators for A, then for any class C we have

$$A \subseteq C \qquad \text{iff} \qquad B \subseteq C \qquad \text{and} \qquad f(A \cap C) \subseteq C.$$

To say that B is a set of generators for A means that $B \subseteq A$ and the only subclass of A which has B as a subclass and is closed under f is A.

Next a class A is defined to be *infinite* if there is a one-to-one mapping $f : A \to A$ such that $f(A) \neq A$. Dedekind notes that the observation of this property of infinite sets is not new, but using it as a definition is new. He goes on to give a proof that there is an infinite class by noting that if s is a thought which he has, then by letting s' be a thought about the thought s he comes to the conclusion that there are an infinite number of possible thoughts, and thus an infinite class of objects.

A is said to be *simply infinite* if there is a one-to-one mapping $f : A \to A$ such that $A \setminus f(A)$ has a single element a in it, and a generates A. He shows that every infinite A has a simply infinite B in it. Combining this with his proof that infinite classes exist we have a proof that simply infinite sets exist. Any two simply infinite classes are shown to be isomorphic, so he says by *abstracting* from simply infinite classes one obtains the *natural numbers* N.

Let 1 be the initial natural number (which generates N), and let n' be the successor of a natural number n (i.e., n' is just $f(n)$). The *ordering* $<$ of the natural numbers is defined by $m < n$ iff the class of elements generated by n is a subclass of the class of elements generated by m'; and the *linearity* of the ordering is proved. Next he introduces *definition by recursion* namely given any set A and any function $\theta : A \to A$ and given any $a \in A$ he proves there is a unique function satisfying the conditions

- $f(1) = a$
- $f(n') = \theta(f(n))$.

He proves this by first showing (by induction) that for each natural number m there is a unique f_m from N_m to A, where N_m is the set $\{n \in N : n \leq m\}$, which satisfies

- $f_m(1) = a$
- $f_m(n') = \theta(f_m(n))$ for $n < m$.

Then he defines

- $f(m) = f_m(m)$.

Now he turns to the definition of the basic operations. For each integer m he uses recursion to get a function $g_m : N \to N$ such that

- $g_m(1) = m'$
- $g_m(n') = (g_m(n))'$.

Then $+$ is defined by

- $m + n = g_m(n)$.

The operation $+$ is then proved to be completely characterized by the following:

- $x + 1 \approx x'$
- $x + y' \approx (x + y)'$.

Likewise multiplication and exponentiation are defined and shown to be characterized by

- $x \times 1 \approx x$
- $x \times y' \approx (x \times y) + x$

- $x^1 \approx x$
- $x^{y'} \approx (x^y) \times x$.

Using induction the following laws are established:

- $x + y \approx y + x$

- $x + (y + z) \approx (x + y) + z$
- $x \times y \approx y \times x$
- $x \times (y \times z) \approx (x \times y) \times z$
- $x \times (y + z) \approx (x \times y) + (x \times z)$
- $(x \times y)^z \approx (x^z) \times (y^z)$
- $x^{y+z} \approx x^y \times x^z$
- $(x^y)^z \approx x^{y \times z}$.

The verification of these fundamental laws can be found in Appendix B of **LMCS**.

Now one can use the usual operations of $+$ and $\times$ on N and the ordering $\leq$ to define their extension first to the integers, then to the rationals, and finally to the reals. Consequently the basic study of the real line has been reduced to the study of natural numbers.