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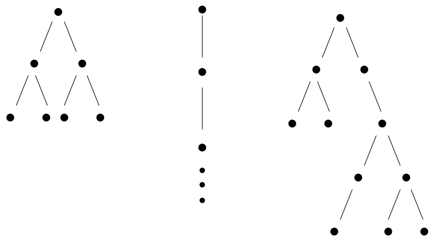
1 The compactness theorem for Propositional Logic

We give a second proof of the important compactness result, Theorem II.8.1 of **LMCS**, using the visual aid of *trees*.

1.1 Binary trees

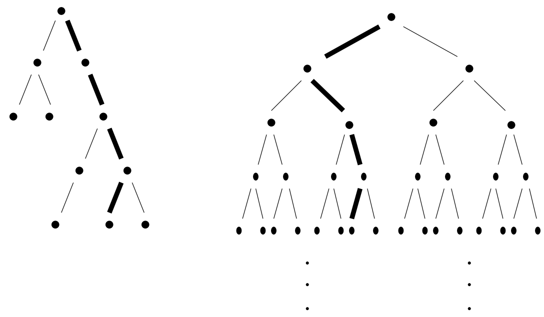
As background to the proof of the compactness theorem we want to introduce the reader to basic notions regarding trees. A *tree* is essentially an upside-down simplified representation of an ordinary tree, e.g.,



Some Examples of trees

The key ingredients of a tree are the *nodes*, indicated in the above figure by solid black circles, and the *edges* connecting them. The node at the top is called the *root* of the tree, and those connected by an edge are said to be *adjacent*. Every node which is not the root has exactly one *adjacent* node above it. A tree is *binary* if every node has at most two adjacent nodes below it. The above examples of trees are binary.

A *branch* in a binary tree is a path starting at the root and proceeding downward along the edges of the tree, and not stopping unless it comes to a node with no adjacent nodes below it. We give two examples below, the first being a finite branch in a finite binary tree, the second is meant to indicate an infinite branch in an infinite binary tree.



Branches in a Tree

The *length* of a branch is the number of edges in the branch, which is one less than the number of nodes in the branch. A branch is *infinite* if it has an infinite number of nodes in it. A tree has *arbitrarily long* branches if for every positive integer n there is a branch in the tree with length at least n. Clearly a tree with an infinite branch has arbitrarily long branches. In the preceding figure we note that the first binary tree is finite and does not have arbitrarily long branches; whereas the second binary tree is infinite, has arbitrarily long branches, and has infinite branches. The following special case of a famous theorem, called König's Lemma, guarantees an infinite branch for certain binary trees.

LEMMA 1 A binary tree with arbitrarily long branches has an infinite branch.

Proof. The idea is simply that if there are arbitrarily long branches passing through a given node a of a binary tree, then for some node a' adjacent to a and immediately below it one must have arbitrarily long branches passing through a'.

To see this consider the two cases, namely we have one node a_1 or two nodes a_1, a_2 adjacent to and below a. In the first case clearly all the branches through a go through a_1, and we are finished. In the second case, if we have no branches of length n_i going though a_i, $1 \leq i \leq 2$, then there are no branches of length $n = \max(n_1, n_2)$ going through a. But this contradicts our assumption that there are arbitrarily long branches through a.

We can start with the root r and repeatedly apply this observation to obtain an infinite branch $r, r', r'', \ldots$. ■

1.2 Second proof of the compactness theorem

Now we want to apply this lemma to show that a set of propositional formulas is satisfiable iff every finite subset is satisfiable. First we need a technical lemma.

LEMMA 2 Let S be a set of propositional formulas. Then there is a set $S' \subseteq S$ such that

(a) no two formulas in S' are truth equivalent, and

(b) $\mathsf{Sat}(S)$ iff $\mathsf{Sat}(S')$.

Proof

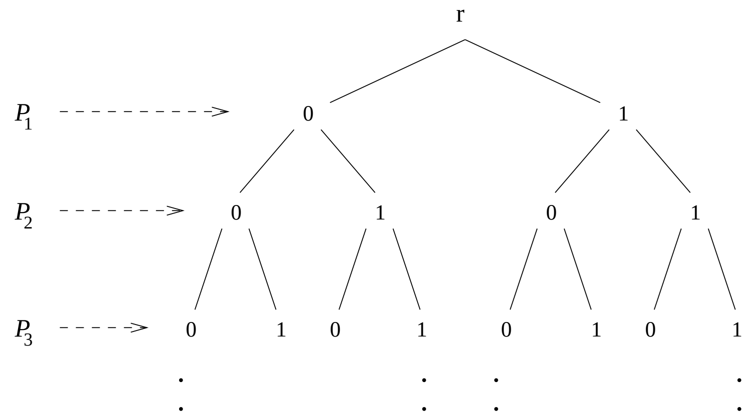
The truth equivalence relation $\sim$ partitions S into equivalence classes. Let S' be a subset of S obtained by choosing exactly one member from each such equivalence class. Then S and S' are satisfied by precisely the same truth evaluations for the variables of S. ■

THEOREM [Compactness for Propositional Logic]
Let S be a set of propositional formulas. Then $\mathrm{Sat}(S)$ iff $\mathrm{Sat}(S_0)$ for every finite $S_0 \subseteq S$.

Proof. First we can use Lemma 2 to assume, without loss of generality, that no two formulas in S are truth equivalent. The direction $(\Longrightarrow)$ is clear, so let us prove $(\Longleftarrow)$. Let S be a set of propositional formulas such that each finite subset of S is satisfiable.

Let $P_1, P_2, \cdots$ be the propositional variables occurring in S; and let S_n be the set of formulas in S which mention only variables from $P_1, \ldots, P_n$. Then S_n has at most 2^{2^n} members (since different members have different truth tables). Thus if only finitely many variables occur in S it follows that S is finite, and therefore satisfiable. So now we assume infinitely many variables occur in S.

A truth evaluation is determined by an assignment of truth values to the propositional variables. Now think of such an assignment as a path through the following binary tree:

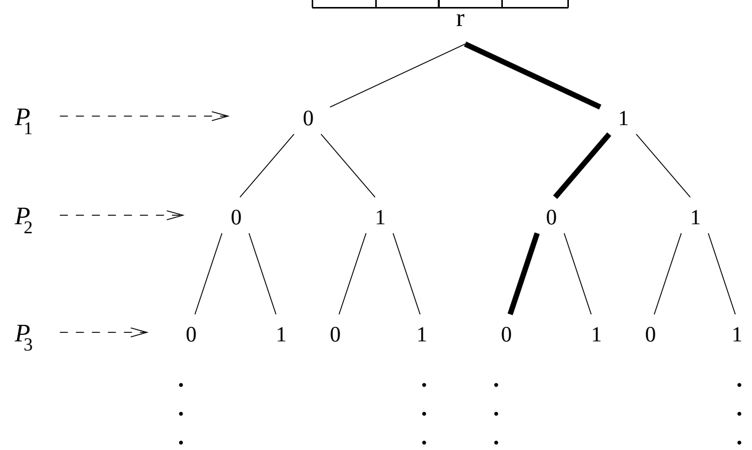


Binary Tree of Truth Values

For example the assignment

P_1	P_2	P_3	$\cdots$
1	0	0	$\cdots$

will give the path



A Path in the Tree

Let T_S be the subtree of this binary tree given by taking all the finite paths $ra_1a_2\cdots a_n$ in the tree such that the truth evaluation $\mathbf{e} = (a_1, \ldots, a_n)$ makes every formula in S_n true. Then clearly T_S is a binary tree, and since one can satisfy any S_n, it must have arbitrarily long branches. Thus, by Lemma 1, the tree T_S must have an infinite branch $ra_1a_2\ldots$. But then the truth evaluation $e_i = a_i$ makes all the formulas in S true, i.e., S is satisfiable. ■