Notes prepared by
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1 Peano

Giuseppe Peano (1858–1932)
1889 - The principles of arithmetic, presented by a new method

A basic education in mathematics will include three references to Peano — his axioms for the natural numbers, his space filling curve, and the solvability of $y' = f(x, y)$ for f continuous. Also his influence on mathematical logic was substantial, largely thanks to his young disciple Bertrand Russell.

Peano's first work on logic (1888) showed that the calculus of classes and the propositional calculus were, up to notation, the same. Next, in *The principles of arithmetic, presented by a new method* (1989), he presented logic and set theoretic notation along with the basic axioms of logic and set theory (including abstraction), and stated his convictions about the possibility of presenting any science in a purely symbolic form. As evidence for this he worked out portions of arithmetic, giving the famous Peano axioms, after stating in the preface:

> In addition the recent work by R. Dedekind *Was sind und was sollen die Zahlen?* (Braunschweig, 1888), in which questions pertaining to the foundations of numbers are acutely examined, was quite useful to me.

As to the nature of his new method we again quote from the preface:

> I have indicated by signs all the ideas which occur in the fundamentals of arithmetic. The signs pertain either to logic or to arithmetic $\cdots$.
>
> I believe, however, that with only these signs of logic the propositions of any science can be expressed, so long as the signs which represent the entities of the science are added.

He starts off with the natural numbers N and the successor function given. His axioms are

- 1 is not the successor of any number Peano's Axioms

- if $m' = n'$ then $m = n$
- (induction) if $X \subseteq N$ is closed under successor, and if $1 \in X$, then $X = N$.

Peano skipped over any attempt to define the natural numbers in logic, thus bypassing certain philosophical issues that mathematicians tend to view as being incapable of precise formulation, and concentrated on the manipulation of symbols, something mathematicians find most agreeable.

Thus we see that Peano's axioms are Dedekind's theorems.[1] This approach would be given its most popular form in Landau's *Grundlagen der Analysis*, 1930, (an excellent book for beginning mathematical German), starting with the set of natural numbers N with a successor function obeying the Peano axioms and proceeding to develop the integers, the rationals, the reals and finally the complex numbers with $+$ and $\times$, and proving the basic laws of these operations in 158 pages and 301 propositions.

Peano's axioms, with induction cast in first-order form, and with the recursive definitions of $+$ and $\times$, would form Peano Arithmetic (PA), a popular subject of mathematical logic. In particular one could derive all known theorems of number theory[2] which could be written in first-order form from Peano Arithmetic; finally, in the mid 1970's Paris & Harrington found a 'natural' example of a first-order number theoretic statement which was true, but could not be derived from PA.

The ambitious *Formulario* project was announced in 1892, the goal being to translate mathematics into Peano's concise and elegant notation. The first edition of this work appeared in 1895, the fifth in 1908. The latter was nearly 500 pages, covering approximately 4,200 theorems on arithmetic, algebra, geometry, limits, differential calculus, integral calculus, and the theory of curves. One could well imagine the satisfaction Peano would enjoy today as director of a mathematics database project.

In 1896 Frege communicated his criticism of Peano's foundations — in particular the lack of clearly stated rules of inference. He doubted that Peano's system could do more than express mathematical theorems. Peano's

[1] The subtle point of first showing that the f_m's exist before defining addition on the natural numbers was overlooked by Peano, and later by Landau who was following Peano (who had defined $\leq$ *after* defining $+$). Grundjot discovered this flaw in Landau's work, and repairs were made following ideas of Kalmar.

[2] Of course Gödel had found a statement in first-order number theory which could not be derived from PA, but it was not the sort of statement one would encounter in traditional number theory.

response was that the ability to give brief and precise form to mathematical theorems would make the importance of his work clear.

Aside from his catalytic influence on Russell we can see that Peano's main contributions to the foundations of mathematics were

i. An elegant notation, which has strongly influenced the symbols used today (e.g. $\cup$, $\cap$, $\subset$, $\supset$, $\in$, and $\emptyset$),

ii. adopting the axiomatic approach to all mathematics (not getting involved in the origins of numbers, etc.), and

iii. the belief that his formalization of logic would suffice for expressing the theorems of any field of science once the symbols appropriate to that field were added.

It is surprising to realize that Peano was the first to introduce distinct notation for *subset of* and *belongs to.*

References

[1] G. Peano, Principles of Arithmetic, Presented by a New Method. 1889. [transl. in *From Frege to Gödel*, van Heijenoort, Harvard Univ. Press, 1971.]