1 Arithmetic II

We return to the two first-order formalizations of arithmetic to make some remarks on how the study of these subjects can be developed further.

1.1 Peano Arithmetic: Definability

Since First-order Arithmetic is known to be rather intractable (it is undecidable), when looking for an effective approach to number theory, something which one might use for an automated theorem prover, it is natural to consider Peano Arithmetic since, as we said, all standard theorems of number theory which can be formulated as first-order statements can be derived from PA. To see that this is true one needs to return to the formulations in the previous section and show that they can be used in PA.

DEFINITION 1

(a) A relation $r \subseteq \omega^n$ is *defined* in PA by the formula $\varphi(x_1, \ldots, x_n)$ if for each $(k_1, \ldots, k_n) \in \omega^n$ we have

$$(k_1, \ldots, k_n) \in r \qquad \Longrightarrow \qquad \text{PA} \vdash \varphi(\bar{k}_1, \ldots, \bar{k}_n)$$
$$(k_1, \ldots, k_n) \notin r \qquad \Longrightarrow \qquad \text{PA} \vdash \neg\varphi(\bar{k}_1, \ldots, \bar{k}_n)$$

(b) A function $f : \omega^n \Longrightarrow \omega$ is *defined* in PA by the formula $\varphi(x_1, \ldots, x_n, y)$ if for each $(k_1, \ldots, k_n, k) \in \omega^{n+1}$ we have

$$f(k_1, \ldots, k_n) = k \qquad \Longrightarrow \qquad \text{PA} \vdash \varphi(\bar{k}_1, \ldots, \bar{k}_n, \bar{k})$$
$$f(k_1, \ldots, k_n) \neq k \qquad \Longrightarrow \qquad \text{PA} \vdash \neg\varphi(\bar{k}_1, \ldots, \bar{k}_n, \bar{k})$$
$$\text{and} \qquad \qquad \text{PA} \vdash \exists! y\, \varphi(\bar{k}_1, \ldots, \bar{k}_n, y)$$

In his 1931 paper Gödel showed that the relations and functions discussed in the previous section on first-order arithmetic are indeed definable in PA[1]. Using this expressive power of PA Gödel went on to give explicit first-order

---

[1] As are all decidable relations and computable functions.

sentences which are true in $\boldsymbol{\omega}$ but cannot be derived from PA.

Problem 1

(a) Show $<$ and $\leq$ are definable in PA.

(b) Show $|$ is definable in PA.

(c) Show $prime(x)$ is definable in PA.

(d) Show $coprime(x)$ is definable in PA.

(e) Show Gödel's β function is definable in PA.

(f) Show x^y is definable in PA.

1.2 The Two Arithmetics

Let us briefly review the two versions of the natural numbers which we have considered.

First-Order Arithmetic	
• **Symbols** $\wedge, \vee, \neg, \implies, \forall, \exists, +, \times, 0, 1$ and variables • **Formulas** first-order formulas	
TRUTH $\models \varphi$ means φ is true of the nonnegative numbers	DERIVATION $\vdash \varphi$
AXIOMS	RULES OF INFERENCE
nothing reasonable exists	

We know from the work of Gödel (1931), Church and Rosser (1936) that we cannot hope for a *reasonable*[2] sound and complete logic to derive all the first-order truths of number theory. But if we are willing to settle for *some*

---

[2]*reasonable* roughly means that we want to be able to generate the axioms and rules of inference.

of the truths of number theory we can try PA, the first-order formulation of Peano's axioms. Then indeed we have a logic system

Peano Arithmetic	
• **Symbols** $\wedge, \vee, \neg, \implies, \forall, \exists, +, \times, 0, 1$ and variables • **Formulas** first-order formulas	
TRUTH $\mathrm{PA} \models \varphi$ means φ is true in any structure in which PA is true	DERIVATION $\mathrm{PA} \vdash \varphi$
AXIOMS PA	RULES OF INFERENCE usual rules for first-order logic

which is sound and complete (provided it is consistent); and all known theorems of traditional number theory which can be expressed in first-order form can be derived from this system. There are some very interesting open problems which can be expressed in first-order arithmetic, e.g., the Riemann Hypothesis. We do not know if these questions can be settled by Peano Arithmetic — the completeness is with respect to all models of PA, not just the nonnegative numbers.

References

[1] A. Church, An unsolvable problem of elementary number theory. The American J. of Math. **58** (1936), 345–363.

[2] K. Gödel, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter System. I. Monatshefte für Mathematik und Physik, **38** (1931), f173–198. [transl. in *From Frege to Gödel*, van Heijenoort, Harvard Univ. Press, 1971.]

[3] J.B. Rosser, Extensions of some theorems of Gödel and Church. J. Symbolic Logic **1** (1936), 87–91.