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A proof of Boole's Expansion Theorem

[THEOREM I.4.3 OF **LMCS**]

Given classes $\vec{A} = A_1, \ldots, A_m$ we say K is an $\vec{A}$-*constituent* if it is a constituent for the classes $A_1, \ldots, A_m$. Boole used the following theorem to *expand* a given expression $F(A_1, \ldots, A_m)$ into its constituents. It says basically that F is a union of $\vec{A}$-constituents, and a given $\vec{A}$-constituent K is a constituent of this expansion of F if and only if $F(\sigma_K) \approx 1$, where σ_K is the unique sequence σ of 1's and 0's that make $K(\sigma) \approx 1$. The following gives the σ_K for the constituents belonging to the three classes A, B, C:

K	σ_K
ABC	111
ABC'	110
$AB'C$	101
$AB'C'$	100
$A'BC$	011
$A'BC'$	010
$A'B'C$	001
$A'B'C'$	000

EXAMPLE 1 If $F(A, B, C) = (AB \cup C)'$ and $K = A'BC'$ then $F(\sigma_K) = F(0, 1, 0) = (01 \cup 0)' = 1$; and note that $A'BC'$ is one of the constituents in the expansion of $(AB \cup C)'$.

Note that if K, L are both $\vec{A}$ constituents then

$$K(\sigma_L) = \begin{cases} 1 & \text{if } K = L \\ 0 & \text{otherwise} \end{cases}.$$

THEOREM [Expansion Theorem] Let $F(X_1, \ldots, X_n)$ be given. Then

$$\begin{aligned} & F(X_1, \ldots, X_n) \approx F(1, 1 \ldots, 1) X_1 X_2 \cdots X_n \\ & \cup\ F(0, 1, \ldots, 1) X_1' X_2 \cdots X_n\ \cup\ F(1, 0, \ldots, 1) X_1 X_2' \cdots X_n \\ & \cup\ \cdots\ \cup\ F(0, \ldots, 0, 0) X_1' X_2' \cdots X_n', \end{aligned}$$

or, more briefly,

$$F(X_1, \ldots, X_n) \approx \bigcup_K F(\sigma_K) K,$$

where K ranges over the $\vec{X}$-constituents.

Proof By using the fundamental identities on page 11 one sees that it is possible to express F as a union of constituents. So we write

$$\begin{aligned} &F(X_1, \ldots, X_n) \approx a_1 X_1 X_2 \cdots X_n \ \cup \ a_2 X_1' X_2 \cdots X_n \\ &\cup \ a_3 X_1 X_2' \cdots X_n \ \cup \ a_4 X_1' X_2' \cdots X_n \ \cup \ \cdots, \end{aligned}$$

where each constant a_i is either 0 or 1. We can write this more briefly as follows (using the fact that there are 2^n constituents for n variables)

$$F(\vec{X}) \approx \bigcup_{j=1}^{2^n} a_j K_j,$$

where the K_j are the various $\vec{X}$-constituents.

Now one observes that for σ_i being the unique sequence of 1's and 0's that gives $K_i(\sigma_i) \approx 1$, i.e., $\sigma_i = \sigma_{K_i}$, we have

$$F(\sigma_i) \quad \approx \quad \bigcup_{j=1}^{2^n} a_j K_j(\sigma_i) \approx a_i,$$

as desired. ∎