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A proof of Boole's Elimination Theorem

THEOREM I.4.7 OF LMCS

This theorem is stated on page 24 of **LMCS**. Boole's proofs, as explained in the historical remarks, were pretty 'wild'. Here is a modern version.

THEOREM [Elimination Theorem] The most general equation $F(\vec{B}) \approx 0$ that follows from $E(\vec{A}, \vec{B}) \approx 0$ is obtained by setting

$$F(\vec{B}) = E(1, 1, \dots, 1, \vec{B})E(0, 1, \dots, 1, \vec{B}) \cdots E(0, 0, \dots, 0, \vec{B}),$$

or, more briefly,

$$F(\vec{B}) = \bigcap_{\sigma} E(\sigma, \vec{B}),$$

where σ ranges over sequences of 1's and 0's.

Proof. First we show $F(\vec{B}) \approx 0$, for the stated choice of F . By expanding E about A_1 (see Problem 4.4 on page 25) we have

$$E(\vec{A}, \vec{B}) \approx E(1, \hat{A}, \vec{B})A_1 \cup E(0, \hat{A}, \vec{B})A'_1,$$

where $\hat{A} = A_2 \cdots A_m$. Then from $E(\vec{A}, \vec{B}) \approx 0$ follows

$$\begin{aligned} E(1, \hat{A}, \vec{B})A_1 &\approx 0 \\ E(0, \hat{A}, \vec{B})A'_1 &\approx 0, \end{aligned}$$

and thus

$$\begin{aligned} E(1, \hat{A}, \vec{B})E(0, \hat{A}, \vec{B})A_1 &\approx 0 \\ E(1, \hat{A}, \vec{B})E(0, \hat{A}, \vec{B})A'_1 &\approx 0. \end{aligned}$$

Taking the union of these two gives

$$E(1, \hat{A}, \vec{B})E(0, \hat{A}, \vec{B})(A_1 \cup A'_1) \approx 0,$$

and thus

$$E(1, \hat{A}, \vec{B})E(0, \hat{A}, \vec{B}) \approx 0.$$

Now repeat the above steps, except this time expand about A_2 , using the last equation which has the form $E'(\hat{A}, \vec{B}) \approx 0$, to obtain

$$(E(1, 1, \tilde{A}, \vec{B})E(0, 1, \tilde{A}, \vec{B}))(E(1, 0, \tilde{A}, \vec{B})E(0, 0, \tilde{A}, \vec{B})) \approx 0,$$

where $\tilde{A} = A_3, \dots, A_m$. Continuing one arrives at the desired conclusion that $F(\vec{B})$, as defined above, is 0.

For the converse suppose $H(\vec{B}) \approx 0$ follows from $E(\vec{A}, \vec{B}) \approx 0$. Let $K(\vec{B})$ be any \vec{B} -constituent of H . Then for any \vec{A} -constituent $L(\vec{A})$ we have $L(\vec{A})K(\vec{B})$ is an \vec{A}, \vec{B} -constituent of H . But then, by Theorem 4.5, LK must be an \vec{A}, \vec{B} -constituent of E as $H \approx 0$ follows from $E \approx 0$. Then $E(\sigma_L, \vec{B})$ has K as a \vec{B} -constituent, i.e., $E(\sigma_L, \vec{B}) \approx K \cup \dots$. So K is a \vec{B} -constituent of F as F is defined to be the intersection over the L 's of the $E(\sigma_L, \vec{B})$'s.

So every \vec{B} -constituent of H is also a \vec{B} -constituent of F . This means, by Theorem 4.5, that $H \approx 0$ follows from $F \approx 0$, so $F \approx 0$ is a more general conclusion than $H \approx 0$. ■