`Notes prepared by`
`Stanley Burris`
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1 Clones

As we have seen in II.5 of **LMCS**, different collections of connectives can have quite different expressive power. An obvious measure of the expressive power of a set of connectives is to take all formulas which can be built out of them, along with their associated truth-tables. However, if we are only interested in the expressive power up to truth-table equivalence, then it suffices just to take the set of truth-tables associated with the formulas. This leads to the notion of a clone.

DEFINITION 1 Given a set A let $\mathcal{F}(\mathrm{A})$ be the set of all constants from A and all finitary functions on A. A *clone* on A is a subset X of $\mathcal{F}(\mathrm{A})$ such that

i. X is closed under composition;

ii. X contains all projection functions;

iii. a constant c is in X implies that finitary functions with constant value c are in X;

iv. if a finitary function with constant value c is in X then the constant c is in X.

Each subset X of $\mathcal{F}(\mathrm{A})$ is contained in a smallest clone on A, which we will call $\mathbf{Cl}(X)$. For an algebra $\mathbf{A}$, the clone generated by the fundamental operations is called the *clone of* $\mathbf{A}$.

Thus, for example, we can now say that a set S of connectives with truth-tables X is adequate for the classical propositional calculus if $\mathbf{Cl}(X) = \mathcal{F}(\{0,1\})$.

A finite algebra $\mathbf{A}$ which has the property that its clone is all functions on A is called a *primal* algebra. In II.5.1 of **LMCS** we essentially pointed out that the two-element Boolean algebra is a primal algebra. One can easily show that the finite fields $\mathbf{GF}(p)$, p a prime, are primal. It is not known if one can test a finite algebra for primality in polynomial time.

DEFINITION 2 Given two sets of connectives S_1, S_2 with associated sets of truth-tables X_1, X_2 we will say that S_1 and S_2 are *truth-table equivalent* if $\mathbf{Cl}(X_1) = \mathbf{Cl}(X_2)$.

EXERCISES

Problem 1 Show that the 3-element Post-algebra $(\{0,1,2\}, \wedge, ')$, where the operations are defined by $i \wedge j = min(i,j)$, and $1' = 2$, $2' = 0$, and $0' = 1$, is primal.

Problem 2 Show a finite field $\mathbf{GF}(p^n)$ is primal iff $n = 1$.

1.1 Duality

We want to define the notion of dual functions for finitary functions on the set {0, 1}. So consider $f : \{0,1\}^n \longrightarrow \{0,1\}$. The *dual* Δf of f is the function obtained by interchanging 1 and 0 in the arguments and values of f, i.e.,

$$\Delta f(x_0, \ldots, x_{n-1}) = \neg f(\neg x_0, \ldots, \neg x_{n-1}).$$

We say that a function is *self-dual* if it is equal to its dual.

Two connectives are said to be dual if their associated truth-tables are dual, and a connective is self-dual if it is its own dual.

For X a set of finitary functions on {0,1} we define ΔX to be the set of duals of members of X.

THEOREM 3 $\mathbf{Cl}(\Delta X) = \Delta\mathbf{Cl}(X)$.

PROOF. It suffices to show that dual and composition commute. But this is easy:

$$\begin{aligned}
& (\Delta f)(\Delta g_0, \ldots, \Delta g_{k-1})(x_0, \ldots, x_{n-1}) \\
= \;& \neg f(\neg(\neg g_0(\neg x_0, \ldots, \neg x_{n-1})), \ldots, \neg(\neg g_{k-1}(\neg x_0, \ldots, x_{n-1}))) \\
= \;& \neg f(g_0(\neg x_0, \ldots, \neg x_{n-1}), \ldots, g_{k-1}(\neg x_0, \ldots, \neg x_{n-1})) \\
= \;& \Delta\left[f(g_0, \ldots, g_{k-1})\right](x_0, \ldots, x_{n-1}).
\end{aligned}$$

REMARK 4

- There are $2^{2^{n-1}}$ self-dual n-ary functions on {0,1} for $n \geq 2$, i.e., $\sqrt{\#n\text{-ary functions}}$ are self-dual.
- The only self-dual binary functions are the projections and their negations.

- The Sheffer stroke and Schröder's connective are dual.
- The connectives $\vee$ and $\wedge$ are dual.
- The connective $\neg$ is self-dual.

1.2 Post's classification of 2-valued logics

Emil Post classified all possible clones on $\{0,1\}$, and hence in a natural sense all possible 2-valued propositional logics. His work was first presented in 1920 as a companion piece to his Ph.D. Thesis,[1] and it was finally published in the book *Two-valued Iterative Systems of Mathematical Logic*,[2] Princeton, 1941. The classification of E. Post was presented in a more modern notation by R. Lyndon (see [3]).

Post's classification of clones on $\{0, 1\}$ consisted in giving a set of generating functions for each clone. One consequence of his classification is that each clone on $\{0, 1\}$ is *finitely generated*. We will give a list of bases for the clones following Lyndon. (Clones come in dual pairs, and only one member of each such pair will be given.)

We will take as our building blocks the constants

$0,\ 1;$

the truth-tables of the standard connectives

$\neg, \vee, \wedge, \implies, \iff;$

the function $+$ defined by

x	y	$x+y$
1	1	0
1	0	1
0	1	1
0	0	0

the function (x, y, z) defined by

$$(x, y, z) = x \wedge (y \vee z);$$

the function $[x, y, z]$ defined by

$$[x, y, z] = x \wedge (y \iff z);$$

[1] His thesis results appear in [4].

[2] In the preface of his book he says that truth-tables were originated by Keyser.

and the functions $d_n(x_0, \ldots, x_{n-1})$, for $n \geq 3$, defined by

$$d_n(x_0, \ldots, x_{n-1}) = \bigvee_{0 \leq i \leq n-1} x_0 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_{n-1},$$

where the notation $\hat{x}_i$ means that x_i is omitted.

Now we are ready to give a basis (up to duality) for each of the countably many clones on {0, 1}:

$\neg$	$\neg, 0$	$\vee$	$\vee, 0$	$\vee, 1$
$\vee, 0, 1$	$\vee, \wedge$	$\vee, \wedge, 0$	$\vee, \wedge, 0, 1$	$+$
$+, \neg$	$\Longrightarrow, \neg$	$\Longrightarrow, \wedge$	$x + y + z$	$x + y + z, \neg$
$\Longrightarrow$	$\Longrightarrow, d_n$	(x, y, z)	$(x, y, z), 0$	$[x, y, z]$
$[x, y, z], \vee$	$(x, y, z), d_n$	$(x, y, z), d_n, 0$	$[x, y, z], d_n$	d_3
$d_3, x + y + z$	$d_3, x + y + z, \neg$	$\emptyset$	1	0,1

Post (1921/1941) suggested that n-valued logics be considered. However one cannot give such a nice classification even for the 3-valued iterative systems since there are continuum many. [See Janov & Mucnik ([2]1959); Hulanicki & Świerczkowski ([1] 1960).] Connectives in n-valued logic can again be thought of as components in circuits. In multi-valued logic one of the interests is to find good components for designing circuits.

Exercises

Problem 3 Determine which binary operations on {0,1} generate the same clone.

Problem 4 Determine which ternary operations generate the same clone on {0,1}.

References

[1] A. Hulanicki and S. Świerckowski, Number of algebras with a given set of elements. Bull. Acad. Polon., Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), 283-284.

[2] Ju.I. Janov and A.A. Muchnik, On the existence of k-valued closed classes that have no bases. (Russian) Dokl. Akad. Nauk SSSR **127** (1959), 44–46.

[3] R. Lyndon, Identities in two-valued calculi. Trans. Amer. Math. Soc. **71** (1951), 457–465.

[4] E. Post, Introduction to a general theory of elementary propositions of logic. Amer. J. Math. **43** (1921), 163-185.