# Outline

# JUSTIFYING BOOLE'S ALGEBRA of LOGIC for CLASSES

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*"The quasimathematical methods of Dr. Boole especially are so magical and abstruse, that they appear to pass beyond the comprehension and criticism of most other writers, and are calmly ignored."*— William Stanley Jevons, 1869, in *The Substitution of Similars*

# THE PARADIGM

| | LOGIC | | ALGEBRA |
|---|---|---|---|
| PREMISSES | Propositions | Translate $\longrightarrow$ | Equations |
| | | | $\downarrow$ |
| CONCLUSIONS | Propositions | Translate $\longleftarrow$ | Equations |

# BOOLE'S STARTING POINT

The development of Boole's algebra of logic is best understood by assuming that he built his system by starting with the symbols and laws of ordinary numerical algebra, which we will take to be the algebra of the integers $\mathbb{Z}$ with binary operation symbols $+, \times, -$.

Then, as discussed in the first lecture, one has clear mathematical grounds (as opposed to fuzzy linguistic grounds) for why his operations $+, -$ must be only partially defined. This is after having defined multiplication to be intersection, 0 to be the empty class, and 1 the universe.

But surely working with the laws of an algebra that Boole knew so well had advantages. He could rapidly try various alternatives, to find what worked.

This allowed him to go further than anyone else before him, achieving the **depth** and **perfection** of his main theorems.

# BOOLE'S PARTIAL ALGEBRAS

$P(U)$ is the collection of subsets of a universe of discourse $U$.

Boole's **operations** and **constants** on $P(U)$ were:

$$\begin{aligned}
\boxed{A \cdot B} \quad &:= \quad A \cap B \\
\boxed{A + B} \quad &:= \quad \begin{cases} A \cup B & \text{when } A \cap B = \emptyset \\ \text{undefined} & \text{when } A \cap B \neq \emptyset \end{cases} \\
\boxed{A - B} \quad &:= \quad \begin{cases} A \setminus B & \text{when } B \subseteq A \\ \text{undefined} & \text{when } B \nsubseteq A \end{cases} \\
\boxed{0} \quad &:= \quad \emptyset \\
\boxed{1} \quad &:= \quad U.
\end{aligned}$$

This gives the **partial algebra** $\mathbf{P}(U) := \langle P(U), +, \cdot, -, 0, 1 \rangle$.

# BOOLE'S TRANSLATIONS

There are four kinds of Aristotelian categorical propositions:

(A) All $x$ is $y$. (E) No $x$ is $y$.

(I) Some $x$ is $y$. (O) Some $x$ is not $y$.

Boole used the following equational translations:

| | 1847 | 1854 |
|---|---|---|
| (A) | $x = xy$ | $x = vy$ |
| (E) | $xy = 0$ | $x = v(1 - y)$ |
| (I) | $v = xy$ | $vx = vy$ |
| (O) | $v = x(1 - y)$ | $vx = v(1 - y)$ |

Now we need to describe the laws and rules of inference for his algebra of classes.

# BOOLE'S LAWS, RULES OF INFERENCE

Boole's **Laws** were the following:

$\boxed{x \cdot y = y \cdot x}$ $\boxed{x + y = y + x}$

$\boxed{x \cdot (y \pm z) = x \cdot y \ \pm \ x \cdot z}$

$\boxed{\text{transposition}}$ $\boxed{x^2 = x}$ (applies only to variables)

$\boxed{0 \cdot y = 0}$ $\boxed{1 \cdot y = y}$

$\boxed{\text{a rearranged product equals the product}}$

Boole's **rules of inference** were:

$\boxed{\text{Equals} \ \square \ \text{Equals gives Equals}}$, for $\square \in \{+, \cdot, -\}$.

# CHARACTERISTIC FUNCTIONS and BOOLE'S ALGEBRAS

1933, Hassler Whitney, *Characteristic functions and the algebra of logic*.

One can reduce the (modern) calculus of classes on the universe $U$ to ordinary numerical algebra using the **characteristic-function** map $A \mapsto \widehat{A}$, where

$$\widehat{A}(u) = \begin{cases} 1 & \text{if } u \in A \\ 0 & \text{if } u \in U \smallsetminus A. \end{cases}$$

We have

$$\emptyset \mapsto 0, \quad U \mapsto 1, \quad A' \mapsto 1 - \widehat{A}$$

$$A \cap B \mapsto \widehat{A} \cdot \widehat{B}$$

$$A \cup B \mapsto \widehat{A} + \widehat{B} - \widehat{A} \cdot \widehat{B}$$

$$A \triangle B \mapsto \widehat{A} + \widehat{B} - 2\widehat{A} \cdot \widehat{B}$$

# EXAMPLE OF WHITNEY'S REDUCTION

Let's use Whitney's reduction to prove the **absorption law**

$$A = A \cup (A \cap B)$$

Proof:

$$A \mapsto \widehat{A}$$

$$A \cup (A \cap B) \ \mapsto \ \widehat{A} \ + \ \widehat{A} \cdot \widehat{B} \ - \ \widehat{A} \cdot (\widehat{A} \cdot \widehat{B}) \ = \ \widehat{A}$$

Since the characteristic function map is 1–1 on $P(U)$, we have the desired conclusion.

# $\mathbf{P}(U)$ and SIGNED MULTISETS

Theodore Hailperin, in *Boole's Logic and Probability*, 1976/1986

made the KEY OBSERVATION:

the characteristic-function mapping $A \mapsto \widehat{A}$

**embeds Boole's partial algebra $\mathbf{P}(U)$**

into the ring $\mathbb{Z}^U$ of functions from $U$ to $\mathbb{Z}$

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The elements of $\mathbb{Z}^U$ are called **signed multisets**.

The idempotent elements of $\mathbb{Z}^U$ are the characteristic functions.

Let $\mathrm{Idemp}(\mathbb{Z}^U)$ be the **idempotent fragment** of $\mathbb{Z}^U$, a partial algebra.

Then $\boxed{\mathbf{P}(U) \cong \mathrm{Idemp}(\mathbb{Z}^U)}$ under the mapping $A \mapsto \widehat{A}$

The uninterpretables of Boole's system became interpretable in the rings $\mathbb{Z}^U$.

# HAILPERIN'S COMPLETION OF BOOLE'S LAWS

Let $\boxed{\mathsf{CR}_1}$ be a set of equational laws for **commutative rings** with **unity**.

Let $\boxed{\mathsf{N}}$ be the set of laws $\boxed{nx = 0 \rightarrow x = 0}$ for $n \geqslant 1$,

that is, the **no-additive-nilpotents** laws. Then define

$$\boxed{\mathsf{NCR}_1 \ := \ \mathsf{CR}_1 \cup \mathsf{N}}\,, \qquad \boxed{{\mathsf{NCR}_1}^+ \ := \ \mathsf{NCR}_1 \cup \{0 \neq 1\}}.$$

Note: $\boxed{\mathbb{Z}^U \text{ satisfies } \mathsf{NCR}_1}$, and $\boxed{\mathbb{Z}^U \text{ satisfies } {\mathsf{NCR}_1}^+}$ for $U \neq \emptyset$

**Hailperin's laws** were ${\mathsf{NCR}_1}^+$ augmented by idempotent Boolean variables (Boole's $x^2 = x$ law).

Hailperin used first-order logic with these laws to derive Boole's four main theorems.

# THE RULE OF 0 AND 1

Boole's **Rule of 0 and 1** said that the axioms, laws and processes of his **algebra of logic** were the same as those of the **algebra of numbers** where the variables were only allowed to take the values 0 and 1.

This remarkable foundation of Boole's algebra of logic was, to my knowledge, almost **totally neglected** until recent years.

Ca. 2000 this was finally understood to mean that an equational argument

$$\boxed{\varepsilon_1(\mathbf{x}), \ldots, \varepsilon_k(\mathbf{x}) \ \therefore \ \varepsilon(\mathbf{x})}$$

was correct in Boole's algebra of logic iff

$$\boxed{\mathbb{Z} \ \models_{01} \ \varepsilon_1(\mathbf{x}) \wedge \cdots \wedge \varepsilon_k(\mathbf{x}) \ \rightarrow \ \varepsilon(\mathbf{x})}$$

where $\models_{01}$ means the variables are restricted to the values 0 and 1.

# EXAMPLES for R01

1. $\boxed{\mathbb{Z} \models_{01} x + y = y + x}$ since

| $x$ | $y$ | $x+y$ | $y+x$ |
|---|---|---|---|
| 1 | 1 | 2 | 2 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 |

2. $\boxed{\mathbb{Z} \models_{01} s(\mathbf{x}) + t(\mathbf{x}) = t(\mathbf{x}) + s(\mathbf{x})}$ (+ is commutative in $\mathbb{Z}$).

3. $\boxed{\mathbb{Z} \models_{01} x^2 = x}$ since $x^2 = x$ for $x \in \{0, 1\}$.

4. $\boxed{\mathbb{Z} \not\models_{01} (x + y)^2 = x + y}$ since $(1 + 1)^2 \neq 1 + 1$, i.e., $4 \neq 2$.

5. $\boxed{\mathbb{Z} \models_{01} 2xy = 0 \rightarrow xy = 0}$

# MAKING R01 PRECISE (Weak Version)

In the following, $\boxed{\mathrm{Idemp}(\mathbf{x})}$ is an abbreviation for

$$(x_1^2 = x_1) \wedge \cdots \wedge (x_m^2 = x_m).$$

Let $\boxed{q(\mathbf{x})}$ be a quasi-equation $\varepsilon_1(\mathbf{x}) \wedge \cdots \wedge \varepsilon_k(\mathbf{x}) \ \rightarrow \ \varepsilon(\mathbf{x})$.

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THEOREM. $\boxed{\mathrm{NCR}_1 \vdash \mathrm{Idemp}(\mathbf{x}) \rightarrow \ q(\mathbf{x})}$ iff $\boxed{\mathbb{Z} \ \models_{01} \ q(\mathbf{x})}$

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PROOF (Sketch). Let $\mathfrak{R}$ be the collection of rings satisfying $\mathrm{NCR}_1$.

Let $\mathfrak{R}_0$ be the members of $\mathfrak{R}$ generated by their idempotent elements. Then the following are equivalent:

(1) $\mathrm{NCR}_1 \vdash \ \mathrm{Idemp}(\mathbf{x}) \rightarrow \ q(\mathbf{x})$ (2) $\mathfrak{R} \ \models \ \mathrm{Idemp}(\mathbf{x}) \rightarrow \ q(\mathbf{x})$

(3) $\mathfrak{R}_0 \ \models \ \mathrm{Idemp}(\mathbf{x}) \rightarrow \ q(\mathbf{x})$ (4) $\mathbb{Z} \ \models \ \mathrm{Idemp}(\mathbf{x}) \rightarrow \ q(\mathbf{x})$.

One can use the **Pierce Sheaf** to show (3) $\Leftrightarrow$ (4).

# HORN FORMULAS

We strengthen the Rule of 0 and 1 using Horn formulas.

A **basic formula** is either an atomic formula or a negated atomic formula.

A **Horn clause** is a disjunction of one or more basic formulas, with at most one of the basic formulas being atomic.

A **Horn formula** $\varphi(\mathbf{x})$ is a first-order formula in prenex form (i.e., quantifiers are in front) whose matrix (i.e., quantifier free part) is a conjunction of one or more Horn clauses.

EXAMPLES: (that are equivalent to Horn formulas)

Equality rules, e.g., $x = y \ \wedge \ y = z \ \rightarrow \ x = z$.

Replacement rules, e.g., $u = v \ \rightarrow \ u + x = v + x$.

Boole's Theorems, e.g., $\bigwedge_i t_i(\mathbf{x}) = 0 \ \longleftrightarrow \ \sum_i t_i(\mathbf{x})^2 \ = \ 0$.

# RELATIVIZING QUANTIFIERS

Given a first-order sentence $\varphi$,

let $\boxed{\varphi|_{\mathsf{Idemp}}}$ be the result of **relativizing the quantifiers** of $\varphi$ to idempotent elements, that is:

Change each subformula $(\forall x)\psi$ to $(\forall x)\big((x^2 = x) \ \rightarrow \ \psi\big)$

Change each subformula $(\exists x)\psi$ to $(\exists x)\big((x^2 = x) \ \wedge \ \psi\big)$.

FACT: If $\varphi$ is equivalent to a Horn sentence, then $\varphi|_{\mathsf{Idemp}}$ is also equivalent to a Horn sentence.

# RULE OF 0 AND 1 (Strong Version)

THEOREM. [B&S, 2013] Given a Horn sentence $\varphi$,

$$\boxed{\mathsf{NCR_1}^+ \vdash \varphi\big|_{\mathsf{Idemp}}} \quad \text{iff} \quad \boxed{\mathbb{Z} \models \varphi\big|_{\mathsf{Idemp}}}.$$

PROOF SKETCH:

Use the same proof as for quasi-equations, adding the observation that, for $\mathbf{R} \in \mathfrak{R}_0$, the Pierce sheaf gives

**R** is isomorphic to a **ring of continuous functions** $C(X, \mathbb{Z})$

from a Boolean space $X$ to the integers $\mathbb{Z}$,

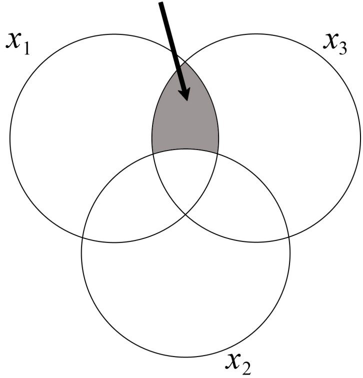
where the integers are given the discrete topology.

$C(X, \mathbb{Z})$ satisfies the Horn theory of $\mathbb{Z}$. [B&W, 1979]

# THE CONSTITUENTS $C_\sigma(\mathbf{x})$ of $\mathbf{x}$

Before stating Boole's main theorems we need the **constituents** $C_\sigma(\mathbf{x})$ of $\mathbf{x}$.

EXAMPLE: $C_{101}(x_1, x_2, x_3) = x_1 \cdot (1 - x_2) \cdot x_3$




They constituents of $\mathbf{x}$ correspond to the regions of a Venn diagram for $\mathbf{x}$.

# BOOLE'S FOUR MAIN THEOREMS:

The strong Rule of 0 and 1 gives quick proofs of Boole's main theorems (see B&S, 2013):

**Expansion** $t(\mathbf{x}, \mathbf{y}) = \sum_\sigma t(\sigma, \mathbf{y}) \cdot C_\sigma(\mathbf{x})$

**Reduction** $\bigwedge_i \big( \; t_i(\mathbf{x}) = 0 \; \big) \; \longleftrightarrow \; 0 = \sum_i t_i(\mathbf{x})^2$

**Elimination** $(\exists \mathbf{x})\big( \; t(\mathbf{x}, \mathbf{y}) = 0 \; \big) \; \longleftrightarrow \; 0 = \prod_\sigma t(\sigma, \mathbf{y})$

**Solution** $\boxed{q(\mathbf{y}) \cdot x = p(\mathbf{y})} \longleftrightarrow$

$$\boxed{p(\mathbf{y}) \cdot \big(p(\mathbf{y}) - q(\mathbf{y})\big) \; = \; 0} \quad \text{and}$$

$$\boxed{(\exists v)\Big(x = \sum_{\substack{\tau \\ p(\tau)=q(\tau)\neq 0}} C_\tau(\mathbf{y}) \; + \; v \cdot \sum_{\substack{\tau \\ p(\tau)=q(\tau)=0}} C_\tau(\mathbf{y}) \; \Big)}$$

# PRINCIPLES OF SYMBOLICAL REASONING

Boole claimed, by his **Principles of Symbolical Reasoning**, that the use of (partially) *uninterpretable* terms was sound when deriving interpretable conclusions from interpretable premisses.

This is false in general. Consider the partial algebra $\mathbf{A} = \langle \{0, 1\}, \star \rangle$ with one binary operation $\star$ with $0 \star 0 = 0$ and $1 \star 1 = 1$; otherwise $\star$ is undefined.

Then we have $x \star y = x$ and $x \star y = y$ both true in $\mathbf{A}$ (whenever both sides are defined). But the consequence $x = y$ does not hold in $\mathbf{A}$.

However Boole's claim works for his partial algebras because of Hailperin's observations.

# RECEPTION of LT (A)

Boole published his last, and most important, work on logic, "Laws of Thought", in 1854.

W.S. JEVONS, a former student of De Morgan, gave the first major response to Boole's system 10 years later.

He found Boole's system—based on partial operations that obeyed the laws of numbers—magical, obscure, etc.

In 1864 he published his book "Pure Logic", based on the total operations of union and intersection, with a limited version of the complement operation.

In 1869/70 he published a paper describing how to make a logic machine to handle arguments in four variables.

# RECEPTION of LT (B)

C.S. PEIRCE , unaware of Jevons priority, published a paper in 1867 using the total operations of union, intersection and complement. His most influential paper was in 1880.

J. VENN was the first person to write a substantial review of Boole's LT—this was in 1876, 22 years after the publication of LT.

Venn was the only major scholar to try to adhere to Boole's system of partial operations.

In 1881 he published his popular book "Symbolic Logic", using Boole's system.

In 1894, in the second edition of "Symbolic Logic", Venn switched to the modern total operations.

# RECEPTION of LT (C)

E. SCHRÖDER was the first person to fully incorporate Boole's four main theorems into the modern total algebra version of the algebra of logic.

This was in his 1877 article "Operationskreis".

He delayed publishing his views on Boole's translations of categorical propositions until his master work "Algebra der Logik" appeared in the 1890s.

There he 'proved' that equations could not be used to express the particular categorical propositions. He said that one needed to use $xy \neq 0$ to express "Some $x$ is $y$".

# RECEPTION of LT (D)

H. WHITNEY came ever so close to having a framework to clarify Boole's system in 1933.

He showed how to translate modern equations (using union, intersection and complement) into numerical equations, by mapping sets $A$ to their characteristic functions $\widehat{A}$.

Then he could use ordinary algebra to verify basic properties of the algebra of sets.

However he failed to realize that characteristic functions, endowed with the operations inherited from $\mathbb{Z}^U$, gave partial algebras isomorphic to Boole's $\mathbf{P}(U)$.

# RESCUING BOOLE'S TRANSLATIONS

Recall:

(A) All $x$ is $y$. (E) No $x$ is $y$.

(I) Some $x$ is $y$. (O) Some $x$ is not $y$.

The equational translations to be used are boxed:

| | 1847 | 1854 | (new) |
|---|---|---|---|
| (A) | $\boxed{x = xy}$ | $x = vy$ | |
| (E) | $\boxed{xy = 0}$ | $x = v(1 - y)$ | |
| (I) | $v = xy$ | $vx = vy$ | $\boxed{v = vxy}$ |
| (O) | $v = x(1 - y)$ | $vx = v(1 - y)$ | $\boxed{v = vx(1 - y)}$ |

The claim is that this translation works for **arguments**, not for individual propositions.

# THANK YOU!

THE END